DISCONTINUOUS FINITE ELEMENT APPROXIMATION OF QUASISTATIC CRACK GROWTH IN FINITE ELASTICITY

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Abstract. We propose a time-space discretization of a general notion of quasistatic growth of brittle fractures in elastic bodies proposed in [13] by G. Dal Maso, G.A. Francfort, and R. Toader, which takes into account body forces and surface loads. We employ adaptive triangulations and prove convergence results for the total, elastic and surface energies. In the case in which the elastic energy is strictly convex, we prove also a convergence result for the deformations.

Keywords : variational models, energy minimization, free discontinuity problems, crack propagation, quasistatic evolution, brittle fracture, finite elements.

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Contents

1. Introduction

The aim of this paper is to provide a discontinuous finite element approximation of a model of quasistatic growth of brittle fractures in finite elasticity recently proposed in [13] by Dal Maso, Francfort, and Toader in the framework of the variational theory of crack propagation proposed by Francfort and Marigo in [15]. This theory is inspired to Griffith's criterion and determines the crack path through a competition between bulk and surface energies.

In the case of linearized elasticity, a first precise mathematical formulation of the model [15] has been given by Dal Maso and Toader [12]: they treat the case of *anti-planar shear* in dimension two assuming that the fractures are compact sets with a finite number of connected components. This analysis has been extended to the case of plane elasticity by Chambolle in [9]. Francfort and Larsen [14], using the framework of SBV functions (see Section 2), proved the existence of a quasistatic growth of brittle fractures in the case of *anti-planar shear* in any dimension $N \geq 2$ and without assumptions on the structure of the fractures which are dealt with the set of jumps of the displacements. Approximation results for the quasistatic evolution of [14] has been given in [16] and in [17] and provide a theoretical basis to the numerical study of the model given in [5].

The quasistatic crack growth proposed by Dal Maso, Francfort, and Toader in [13] consider the case of finite elasticity, and takes into account possible volume and traction forces applied to the elastic body. In order to describe the result of [13] (a complete description is given in Section 3), let us assume that the elastic body has a reference configuration given by $\Omega \subseteq \mathbb{R}^N$ open, bounded and with Lipschitz boundary. Let $\partial_D\Omega \subseteq \partial\Omega$ be open in the relative topology, and let

$\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$. Let $\Omega_B \subseteq \Omega$, and let $\partial_S\Omega \subseteq \partial_N\Omega$ be such that $\overline{\Omega}_B \cap \partial_S\Omega = \emptyset$. Ω_B is the *brittle part* of Ω, and $\partial_S\Omega$ is the part of the boundary where traction forces are supposed to act. A crack is given by any rectifiable set in $\overline{\Omega}_B$ with finite $(N-1)$ Hausdorff measure. Given a boundary deformation g on $\partial_D\Omega$ and a crack Γ, the family of all admissible deformation of Ω is given by the set $AD(g,\Gamma)$ of all function $u \in GSBV(\Omega;\mathbb{R}^N)$ (see Section 2) such that $S(u) \subseteq \Gamma$ and $u = g$ on $\partial_D\Omega \setminus \Gamma$. Here $S(u)$ denotes the set of jumps of u, and the equality $u = g$ is intended in the sense of traces. Requiring $u = g$ only on $\partial_D\Omega \setminus \Gamma$ means that the deformation is assumed not to be transmitted through the fracture. The bulk energy considered in [13] is of the form

$$\int_\Omega W(x,\nabla u(x))\,dx,$$

where $W(x,\xi)$ is quasiconvex in ξ, and satisfies suitable regularity and growth assumptions (see (3.4) and (3.5)). Moreover the time dependent body and traction forces are supposed to be conservative with work given by

$$-\int_{\Omega\setminus\Gamma} F(t,x,u(x))\,dx - \int_{\partial_S\Omega} G(t,x,u(x))\,d\mathcal{H}^{N-1}(x),$$

where F and G satisfy suitable regularity and growth conditions (see Section 3). Finally the work made to produce the crack Γ is given by

$$\mathcal{E}^s(\Gamma) := \int_\Gamma k(x,\nu(x))\,d\mathcal{H}^{N-1}(x),$$

where $\nu(x)$ is the normal to Γ at x, and $k(x,\nu)$ satisfies standard hypotheses which guarantee lower semicontinuity (see Section 3). Clearly, W, F, G and k depend on the material. Let us set

$$\mathcal{E}^{el}(t)(u) := \int_\Omega W(x,\nabla u(x))\,dx - \int_{\Omega\setminus\Gamma} F(t,x,u(x))\,dx - \int_{\partial_S\Omega} G(t,x,u(x))\,d\mathcal{H}^{N-1}(x),$$

and

$$\mathcal{E}(t)(u,\Gamma) := \mathcal{E}^{el}(t)(u) + \mathcal{E}^s(\Gamma). \tag{1.1}$$

Given a boundary deformation $g(t)$ with $t \in [0,T]$ and a preexisting crack Γ_0, a quasistatic crack growth relative to g and Γ_0 is a map $\{t \to (u(t),\Gamma(t)) : t \in [0,T]\}$ such that the following conditions hold:

(1) for all $t \in [0,T]$: $u(t) \in AD(g(t),\Gamma(t))$;

(2) *irreversibility*: $\Gamma_0 \subseteq \Gamma(s) \subseteq \Gamma(t)$ for all $0 \le s \le t \le T$;

(3) *static equilibrium*: for all $t \in [0,T]$ and for all admissible configurations (u,Γ) with $\Gamma(t) \subseteq \Gamma$
$$\mathcal{E}(t)(u(t),\Gamma(t)) \le \mathcal{E}(t)(u,\Gamma);$$

(4) *nondissipativity*: the time derivative of the total energy $\mathcal{E}(t)(u(t),\Gamma(t))$ is equal to the power of external forces (see (3.21)).

In this paper we discretize the model using a suitable finite element method and prove its convergence to this notion of quasistatic crack growth. We restrict our analysis to a two dimensional setting considering only a polygonal reference configuration $\Omega \subseteq \mathbb{R}^2$.

The discretization of the domain Ω is carried out as in [17] employing *adaptive triangulations* introduced by M. Negri in [19] (see also [20]). Let us fix two parameters $\varepsilon > 0$ and $a \in]0, \frac{1}{2}[$. We consider a regular triangulation $\mathbf{R}_\varepsilon$ of size ε of Ω, i.e. we assume that there exist two constants c_1 and c_2 so that every triangle $T \in \mathbf{R}_\varepsilon$ contains a ball of diameter $c_1\varepsilon$ and is contained in a ball of diameter $c_2\varepsilon$. In order to treat the boundary data, we assume also that $\partial_D\Omega$ is composed of edges of $\mathbf{R}_\varepsilon$. On each edge $[x,y]$ of $\mathbf{R}_\varepsilon$ we consider a point z such that $z = tx + (1-t)y$ with $t \in [a, 1-a]$. These points are called *adaptive vertices*. Connecting together the adaptive vertices, we divide every $T \in \mathbf{R}_\varepsilon$ into four triangles. We take the new triangulation $\mathbf{T}$ obtained after this division as the discretization of Ω. The family of all such triangulations will be denoted by $\mathcal{T}_{\varepsilon,a}(\Omega)$.

The discretization of the energy functional is obtained restricting the total energy (1.1) to the family of functions u which are affine on the triangles of some triangulation $\mathbf{T}(u) \in \mathcal{T}_{\varepsilon,a}(\Omega)$ and are allowed to jump across the edges of $\mathbf{T}(u)$ contained in $\overline{\Omega}_B$. We indicate this space by $\mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$. The boundary data is assumed to belong to the space $\mathcal{AF}_\varepsilon(\Omega;\mathbb{R}^2)$ of continuous functions which are affine on every triangle $T \in \mathbf{R}_\varepsilon$.

Given the boundary data $g_\varepsilon \in W^{1,1}([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2))$ with $g_\varepsilon(t) \in \mathcal{AF}_\varepsilon(\Omega;\mathbb{R}^2)$ for all $t \in [0,T]$ (p, q are related to the growth assumptions on W, F, G) and an initial crack $\Gamma^0_{\varepsilon,a}$ (see Section 6), we divide $[0,1]$ into subintervals $[t^\delta_i, t^\delta_{i+1}]$ of size $\delta > 0$ for $i = 0, \dots, N_\delta$, and for all $u \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$ we indicate by $S^{g_\varepsilon(t)}_D(u)$ the edges of the triangulation $\mathbf{T}(u)$ contained in $\partial_D\Omega$ on which $u \neq g_\varepsilon(t)$. Using a variational argument (Proposition 6.1), we construct a *discrete evolution* $\{(u^{\delta,i}_{\varepsilon,a}, \Gamma^{\delta,i}_{\varepsilon,a}) : i = 0, \dots, N_\delta\}$ such that for all $i = 0, \dots, N_\delta$ we have $u^{\delta,i}_{\varepsilon,a} \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$,

$$\Gamma^{\delta,i}_{\varepsilon,a} := \bigcup_{r=0}^{i} \left[S(u^{\delta,r}_{\varepsilon,a}) \cup S^{g_\varepsilon(t^\delta_r)}_D(u^{\delta,r}_{\varepsilon,a}) \right],$$

and the following *unilateral minimality property* holds: for all $v \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$

$$\mathcal{E}^{el}(t^\delta_i)(u^{\delta,i}_{\varepsilon,a}) \leq \mathcal{E}^{el}(t^\delta_i)(v) + \mathcal{E}^s\left(\left(S(v) \cup S^{g_\varepsilon(t^\delta_i)}_D(v) \right) \setminus \Gamma^{\delta,i-1}_{\varepsilon,a} \right). \tag{1.2}$$

Notice that by construction $u^{\delta,i}_{\varepsilon,a} \in AD(g_\varepsilon(t^\delta_i), \Gamma^{\delta,i}_{\varepsilon,a})$. Moreover the definition of the discrete fracture ensures that $\Gamma^{\delta,i}_{\varepsilon,a} \subseteq \Gamma^{\delta,j}_{\varepsilon,a}$ for all $i \leq j$, recovering in this discrete setting the irreversibilty of the crack growth given in (2). The minimality property (1.2) is the reformulation in the finite element space of the equilibrium condition (3). Finally we obtain an estimate from above for $\mathcal{E}(t^\delta_i)(u^{\delta,i}_{\varepsilon,a}, \Gamma^{\delta,i}_{\varepsilon,a})$ (see Proposition 6.2) which is a discrete version of (4).

In order to perform the asymptotic analysis of the *discrete evolution* $\{(u^{\delta,i}_{\varepsilon,a}, \Gamma^{\delta,i}_{\varepsilon,a}) : i = 0, \dots, N_\delta\}$ we make the piecewise constant interpolation in time $u^\delta_{\varepsilon,a}(t) = u^{\delta,i}_{\varepsilon,a}$ and $\Gamma^\delta_{\varepsilon,a}(t) = \Gamma^{\delta,i}_{\varepsilon,a}$ for all $t^\delta_i \leq t < t^\delta_{i+1}$. Let us suppose that

$$g_\varepsilon \to g \qquad \text{strongly in } W^{1,1}([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2))$$

(where on $W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)$ we take the norm $\|u\| := \|u\|_{W^{1,p}(\Omega;\mathbb{R}^2)} + \|u\|_{L^q(\Omega;\mathbb{R}^2)}$), and that $\Gamma^0_{\varepsilon,a}$ approximate an initial crack Γ^0 in the sense of Proposition 5.1.

The main result of the paper (Theorem 7.1) states that there exist a quasistatic evolution $\{t \to (u(t), \Gamma(t)) : t \in [0,T]\}$ in the sense of [13] relative to the boundary deformation g and the preexisting crack Γ^0 and sequences $\delta_n \to 0$, $\varepsilon_n \to 0$, $a_n \to 0$, such that setting

$$u_n(t) := u^{\delta_n}_{\varepsilon_n,a_n}(t), \qquad \Gamma_n(t) := \Gamma^{\delta_n}_{\varepsilon_n,a_n}(t),$$

for all $t \in [0,T]$ the following facts hold:

(a) $(u_n(t))_{n\in\mathbb{N}}$ is weakly precompact in $GSBV^p_q(\Omega;\mathbb{R}^2)$, and every accumulation point $\tilde{u}(t)$ is such that $\tilde{u}(t) \in AD(g(t), \Gamma(t))$, and $(\tilde{u}(t), \Gamma(t))$ satisfy the static equilibrium (2); moreover there exists a subsequence $(\delta_{n_k}, \varepsilon_{n_k}, a_{n_k})_{k\in\mathbb{N}}$ of $(\delta_n, \varepsilon_n, a_n)_{n\in\mathbb{N}}$ (depending on t) such that

$$u_{n_k}(t) \rightharpoonup u(t) \quad \text{ weakly in } GSBV^p_q(\Omega;\mathbb{R}^2)$$

(see Section 2 for a precise definition of $GSBV^p_q(\Omega;\mathbb{R}^2)$, and of weak convergence in this space);

(b) convergence of the total energy holds, and more precisely elastic and surface energies converge separately, that is

$$\mathcal{E}^{el}(t)(u_n(t)) \to \mathcal{E}^{el}(t)(u(t)), \qquad \mathcal{E}^s(\Gamma_n(t)) \to \mathcal{E}^s(\Gamma(t)).$$

By point (a), the approximation of the deformation $u(t)$ is available only up to a subsequence depending on t: this is due to the possible non uniqueness of the minimum energy deformation

associated to $\Gamma(t)$. In the case $\mathcal{E}^{el}(t)(u)$ is strictly convex, it turns out that the deformation $u(t)$ is uniquely determined, and we prove that (Theorem 8.1)

$$\nabla u_n(t) \to \nabla u(t) \qquad \text{strongly in } L^p(\Omega; M^{2\times 2}),$$

and

$$u_n(t) \to u(t) \qquad \text{strongly in } L^q(\Omega; \mathbb{R}^2).$$

The main difficulty to prove Theorem 7.1 consists in passing to the limit in the static equilibrium (1.2). In order to find the fracture $\Gamma(t)$ in the limit, in Lemma 7.2 and Lemma 7.4 we adapt to the context of finite elements the notion of σ^p- convergence of sets formulated in [13]. This is the key tool to obtain the convergence of elastic and surface energies at all times $t \in [0, T]$ (while in [17] this was available only at the continuity points of $\mathcal{H}^1(\Gamma(t))$). In order to infer the static equilibrium of $\Gamma(t)$ from that of $\Gamma_n(t)$, we employ a generalization of the piecewise affine transfer of jumps [17, Proposition 5.1] (see Proposition 4.2).

The paper is organized as follows. In Section 2 we introduce the basic notation, and some tools employed throughout the paper. In Section 3 we describe the quasistatic crack growth of [13] precising the functional setting and the hypotheses on the elastic and surface energies involved. In Section 4 we introduce the finite element space, and in Section 5 we prove an approximation result for a preexisting crack configuration. In Section 6 we prove the existence of a discrete evolution, and in Section 7 we prove the main approximation result (Theorem 7.1). In Section 8 we treat the case of strictly convex total energy.

2. Notations and Preliminaries

In this section we introduce the main notations and the preliminary results employed in the rest of the paper.

Basic notation. We will employ the following basic notation:

- $M^{n\times m}$ is the space of $n \times m$ matrices;
- $\mathcal{H}^1$ is the one-dimensional Hausdorff measure;
- for $p \in [1, +\infty]$, $\|\cdot\|_p$ denotes the usual L^p norm;
- if μ is a measure on $\mathbb{R}^2$ and A is a Borel subset of $\mathbb{R}^2$, $\mu \llcorner A$ denotes the restriction of μ to A, i.e. $(\mu \llcorner A)(B) := \mu(B \cap A)$ for all Borel sets $B \subseteq \mathbb{R}^2$;
- if $A, B \subseteq \mathbb{R}^2$, $A \,\tilde{\subset}\, B$ means that $A \subseteq B$ up to a set of $\mathcal{H}^1$-measure zero.

SBV and GSBV spaces. Let A be an open subset of $\mathbb{R}^n$, and let $u : A \to \mathbb{R}^m$ be a measurable function. Given $x \in A$, we say that $\tilde{u}(x)$ is the *approximate limit* of u at x, and we write $\tilde{u}(x) = \operatorname*{ap\,lim}_{y\to x} u(y)$, if for every $\varepsilon > 0$

$$\lim_{r\to 0} r^{-n} \mathcal{L}^n \left(\{y \in B_r(x) \,:\, |u(y) - \tilde{u}(x)| > \varepsilon\}\right) = 0.$$

Here $B_r(x)$ denotes the ball of center x and radius r. We indicate by $S(u)$ the set of points where the approximate limit of u does not exist. We say that the matrix $m \times n$ $\nabla u(x)$ is the approximate gradient of u at x if

$$\operatorname*{ap\,lim}_{y\to x} \frac{u(y) - u(x) - \nabla u(x)(y - x)}{|y - x|} = 0.$$

We say that $u \in BV(A; \mathbb{R}^m)$ if $u \in L^1(A; \mathbb{R}^m)$, and its distributional derivative Du is a vector-valued Radon measure on A. In this case, it turns out that $S(u)$ is rectifiable, that is there exists a sequence $(M_i)_{i\in\mathbb{N}}$ of C^1-manifolds such that $S(u) \subseteq \bigcup_i M_i$ up to a set of $\mathcal{H}^{n-1}$-measure zero; as a consequence $S(u)$ admits a normal ν_x for $\mathcal{H}^{n-1}$-almost every $x \in S(u)$. Moreover the approximate gradient $\nabla u(x)$ exists for a.e. $x \in A$, and ∇u is the density of the absolutely continuous part of Du.

We say that $u \in SBV(A; \mathbb{R}^m)$ if $u \in BV(A; \mathbb{R}^m)$ and the singular part $D^s u$ of its distributional derivative Du is concentrated on $S(u)$. The space $SBV(A; \mathbb{R}^m)$ is called the space of $\mathbb{R}^m$-valued *special functions of bounded variation.* For more details, the reader is referred to [4]. We indicate

with $SBV_{loc}(A,\mathbb{R}^m)$ the space of functions which belong to $SBV(A',\mathbb{R}^m)$ for every open set A' with compact closure in A.

The set $GSBV(A,\mathbb{R}^m)$ is defined as the set of functions $u : A \to \mathbb{R}^m$ such that $\varphi(u) \in SBV_{loc}(A)$ for every $\varphi \in C^1(\mathbb{R}^m)$ such that the support of $\nabla\varphi$ has compact closure in $\mathbb{R}^m$. If $p \in]1,+\infty[$, we set

$$GSBV^p(A,\mathbb{R}^m) := \{u \in GSBV(A,\mathbb{R}^m) \,:\, \nabla u \in L^p(A, M^{m\times n}),\, \mathcal{H}^{n-1}(S(u)) < +\infty\}.$$

By [13, Proposition 2.2] the space $GSBV^p(A,\mathbb{R}^m)$ coincide with $(GSBV^p(A,\mathbb{R}))^m$, that is $u := (u_1,\dots,u_m) \in GSBV^p(A,\mathbb{R}^m)$ if and only if $u_i \in GSBV^p(A,\mathbb{R})$ for every $i = 1,\dots,m$.

The following compactness and lower semicontinuity result will be used in the following sections. For a proof, we refer to [2].

Theorem 2.1. *Let A be an open and bounded subset of $\mathbb{R}^n$. Let $g(x,u) : A\times\mathbb{R}^m \to [0,\infty]$ be a Borel function, lower semicontinuous in u and satisfying the condition*

$$\lim_{|u|\to\infty} g(x,u) = +\infty \text{ for a.e. } x \in A.$$

Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $GSBV^p(A;\mathbb{R}^m)$ such that

$$\limsup_k \int_A |\nabla u_k(x)|^p\,dx + \mathcal{H}^{n-1}(S(u_k)) + \int_A g(x,u_k(x))\,dx < +\infty.$$

Then there exists a subsequence $(u_{k_h})_{h\in\mathbb{N}}$ and a function $u \in GSBV^p(A;\mathbb{R}^m)$ such that

$$\begin{aligned} u_{k_h} &\to u && \text{in measure,} \\ \nabla u_{k_h} &\rightharpoonup \nabla u && \text{weakly in } L^p(A;M^{m\times n}). \end{aligned} \tag{2.1}$$

Moreover we have that

$$\mathcal{H}^{n-1}(S(u)) \le \liminf_h \mathcal{H}^{n-1}(S(u_{k_h})).$$

Let $q \in]1,+\infty[$ and let us set

$$GSBV^p_q(A;\mathbb{R}^m) := GSBV^p(A;\mathbb{R}^m) \cap L^q(A;\mathbb{R}^m). \tag{2.2}$$

We say that $u_k \rightharpoonup u$ weakly in $GSBV^p_q(A;\mathbb{R}^m)$ if

$$\begin{aligned} u_k &\to u && \text{in measure} \\ \nabla u_k &\rightharpoonup \nabla u && \text{weakly in } L^p(A;M^{m\times n}) \\ u_k &\rightharpoonup u && \text{weakly in } L^q(A;\mathbb{R}^m). \end{aligned} \tag{2.3}$$

We will often use the following fact: if $u_k \rightharpoonup u$ weakly in $GSBV^p_q(A;\mathbb{R}^m)$ and $\Gamma \subseteq A$ is such that $\mathcal{H}^{N-1}(\Gamma) < +\infty$ and $S(u_k) \subseteq \Gamma$ up to a set of $\mathcal{H}^{N-1}$-measure zero for all k, then $S(u) \subseteq \Gamma$ up to a set of $\mathcal{H}^{N-1}$-measure zero.

Γ-convergence. Let us recall the definition of De Giorgi's Γ-*convergence* in metric spaces: we refer the reader to [11] for an exhaustive treatment of this subject. Let (X,d) be a metric space. We say that a sequence $F_h : X \to [-\infty,+\infty]$ Γ-converges to $F : X \to [-\infty,+\infty]$ (as $h \to +\infty$) if for all $u \in X$ we have

(i) (Γ-*liminf inequality*) for every sequence $(u_h)_{h\in\mathbb{N}}$ converging to u in X,

$$\liminf_{h\to+\infty} F_h(u_h) \ge F(u);$$

(ii) (Γ-*limsup inequality*) there exists a sequence $(u_h)_{h\in\mathbb{N}}$ converging to u in X, such that

$$\limsup_{h\to+\infty} F_h(u_h) \le F(u).$$

The function F is called the Γ-limit of (F_h) (with respect to d), and we write $F = \Gamma-\lim_h F_h$.

We say that a family of functionals $\{F_\varepsilon\}$ Γ-converges to F as $\varepsilon \to 0$ if for every sequence $\varepsilon_h \to 0$ as $h \to +\infty$ we have $\Gamma-\lim_h F_{\varepsilon_h} = F$.

The peculiarity of this type of convergence is its variational character explained in the following proposition.

Proposition 2.2. *Assume that the sequence $(F_h)_{h\in\mathbb{N}}$ Γ-converges to F and that there exists a compact set $K \subseteq X$ such that for all $h \in \mathbb{N}$*

$$\inf_{u\in K} F_h(u) = \inf_{u\in X} F_h(u).$$

Then F admits a minimum on X, $\inf_X F_h \to \min_X F$, and any limit point of any sequence $(u_h)_{h\in\mathbb{N}}$ such that

$$\lim_{h\to+\infty}\Big(F_h(u_h) - \inf_{u\in X} F_h(u)\Big) = 0$$

is a minimizer of F.

Hausdorff metric on compact sets. Let $A \subseteq \mathbb{R}^2$ be open and bounded, and let $\mathcal{K}(\overline{A})$ be the set of all compact subsets of $\overline{A}$. $\mathcal{K}(\overline{A})$ can be endowed by the Hausdorff metric d_H defined by

$$d_H(K_1, K_2) := \max\left\{\sup_{x\in K_1}\operatorname{dist}(x, K_2), \sup_{y\in K_2}\operatorname{dist}(y, K_1)\right\},$$

with the conventions $\operatorname{dist}(x, \emptyset) = \operatorname{diam}(A)$ and $\sup\emptyset = 0$, so that $d_H(\emptyset, K) = 0$ if $K = \emptyset$ and $d_H(\emptyset, K) = \operatorname{diam}(A)$ if $K \neq \emptyset$. It turns out that $\mathcal{K}(\overline{A})$ endowed with the Hausdorff metric is a compact space (see e.g. [21]).

3. The quasistatic crack growth of Dal Maso-Francfort-Toader

In this section we describe the quasistatic evolution of brittle fractures proposed in [13]. They consider the case of n-dimensional finite elasticity, for an arbitrary $n \geq 1$, with a quasiconvex bulk energy and with prescribed boundary deformations and applied loads, depending on time. Since we are going to approximate the case $n = 2$, we prefer to introduce the model in this particular case. For more details, we refer the reader to [13].

Let Ω be a bounded open set of $\mathbb{R}^2$ with Lipschitz boundary and let Ω_B be an open subset of Ω. Let $\partial_N\Omega \subseteq \partial\Omega$ be closed in the relative topology, and let $\partial_D\Omega := \partial\Omega \setminus \partial_N\Omega$. Let $\partial_S\Omega \subseteq \partial_N\Omega$ be closed in the relative topology and such that $\overline{\Omega}_B \cap \partial_S\Omega = \emptyset$. In the model proposed in [13], Ω_B represents the brittle part of Ω, and $\partial_D\Omega$ the part of the boundary on which the deformation is prescribed. Moreover the elastic body Ω is supposed to be subject to surface forces acting on $\partial_S\Omega$.

Admissible cracks and deformations. The set of admissible cracks is given by

$$\mathcal{R}(\overline{\Omega}_B; \partial_N\Omega) := \{\Gamma : \Gamma \text{ is rectifiable }, \Gamma \,\tilde{\subset}\, (\overline{\Omega}_B \setminus \partial_N\Omega),\, \mathcal{H}^1(\Gamma) < +\infty\}.$$

Here $A \,\tilde{\subset}\, B$ means that $A \subseteq B$ up to a set of $\mathcal{H}^1$-measure zero, and Γ rectifiable means that there exists a sequence (M_i) of C^1-manifolds such that $\Gamma \,\tilde{\subset}\, \bigcup_i M_i$. If Γ is rectifiable, we can define normal vector fields ν to Γ in the following way: if $\Gamma = \bigcup_i \Gamma_i$ with $\Gamma_i \,\tilde{\subset}\, M_i$ and $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$, given $x \in \Gamma_i$, we take $\nu(x) = \nu_{M_i}(x)$, where $\nu_{M_i}(x)$ is a normal vector to the C^1-manifold M_i at x. It turns out that two normal vector fields associated to different decompositions $\bigcup_i \Gamma_i$ of Γ coincide up to the sign $\mathcal{H}^1$ almost everywhere.

Given a crack Γ, an admissible deformation is given by any function $u \in GSBV(\Omega; \mathbb{R}^2)$ such that $S(u) \,\tilde{\subset}\, \Gamma$.

The surface energy. The surface energy of a crack Γ is given by

$$\mathcal{E}^s(\Gamma) := \int_\Gamma k(x, \nu(x))\, d\mathcal{H}^1(x), \tag{3.1}$$

where ν is a unit normal vector field on Γ. Here $k : \overline{\Omega}_B \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $k(x, \cdot)$ is a norm in $\mathbb{R}^2$ for all $x \in \overline{\Omega}_B$ and for all $x \in \overline{\Omega}_B$ and $\nu \in \mathbb{R}^2$

$$K_1|\nu| \le k(x, \nu) \le K_2|\nu|, \tag{3.2}$$

where $K_1, K_2 > 0$. Notice that since k is even in the second variable, we have that the integral (3.1) is independent of the orientation given to Γ, that is independent of the particular choice of the unit normal vector field ν.

The bulk energy. Let $p > 1$ be fixed. Given a deformation $u \in GSBV^p(\Omega; \mathbb{R}^2)$ the associated *bulk energy* is given by

$$\mathcal{W}(\nabla u) := \int_\Omega W(x, \nabla u(x))\, dx, \tag{3.3}$$

where $W : \Omega \times M^{2\times 2} \to [0, +\infty)$ is a Carathéodory function satisfying

$$\text{for every } x \in \Omega : \ W(x, \cdot) \text{ is quasiconvex and } C^1 \text{ on } M^{2\times 2}, \tag{3.4}$$

$$\text{for every } (x, \xi) \in \Omega \times M^{2\times 2} : a_0^W |\xi|^p - b_0^W(x) \le W(x, \xi) \le a_1^W |\xi|^p + b_1^W(x). \tag{3.5}$$

Here $a_0^W, a_1^W > 0$, and $b_0^W, b_1^W \in L^1(\Omega)$ are nonnegative functions. Quasiconvexity of W means that for all $\xi \in M^{2\times 2}$ and for all $\varphi \in C_c^\infty(\Omega; \mathbb{R}^2)$

$$W(\xi) \le \int_\Omega W(\xi + \nabla\varphi)\, dx.$$

If we denote by $\partial_\xi W : \Omega \times M^{2\times 2} \to M^{2\times 2}$ the partial derivative of W with respect to ξ, it turns out that there exists a positive constant $a_2^W > 0$ and a nonnegative function $b_2^W \in L^{p'}(\Omega)$, with $p' := p/(p-1)$, such that for all $(x, \xi) \in \Omega \times M^{2\times 2}$

$$|\partial_\xi W(x, \xi)| \le a_2^W |\xi|^{p-1} + b_2^W(x). \tag{3.6}$$

By (3.5) and (3.6) the functional $\mathcal{W}$, defined for all $\Phi \in L^p(\Omega; M^{2\times 2})$ by

$$\mathcal{W}(\Phi) := \int_\Omega W(x, \Phi(x))\, dx,$$

is of class C^1 on $L^p(\Omega; M^{2\times 2})$, and its differential $\partial\mathcal{W} : L^p(\Omega; M^{2\times 2}) \to L^{p'}(\Omega; M^{2\times 2})$ is given by

$$\langle \partial\mathcal{W}(\Phi), \Psi\rangle = \int_\Omega \partial_\xi W(x, \Phi(x))\Psi(x)\, dx, \qquad \Phi, \Psi \in L^p(\Omega; M^{2\times 2}),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between the spaces $L^{p'}(\Omega; M^{2\times 2})$ and $L^p(\Omega; M^{2\times 2})$. By (3.5) and (3.6), there exist six positive constants $\alpha_0^{\mathcal{W}} > 0$, $\alpha_1^{\mathcal{W}} > 0$, $\alpha_2^{\mathcal{W}} > 0$, $\beta_0^{\mathcal{W}} \ge 0$, $\beta_1^{\mathcal{W}} \ge 0$, $\beta_2^{\mathcal{W}} \ge 0$ such that for every Φ, $\Psi \in L^p(\Omega; M^{2\times 2})$

$$\alpha_0^{\mathcal{W}} \|\Phi\|_p^p - \beta_0^{\mathcal{W}} \le \mathcal{W}(\Phi) \le \alpha_1^{\mathcal{W}} \|\Phi\|_p^p + \beta_1^{\mathcal{W}},$$

$$|\langle \partial\mathcal{W}(\Phi), \Psi\rangle| \le (\alpha_2^{\mathcal{W}} \|\Phi\|_p^{p-1} + \beta_2^{\mathcal{W}})\|\Psi\|_p. \tag{3.7}$$

The body forces. Let $q > 1$ be fixed. The density of applied body forces per unit volume in the reference configuration relative to the deformation u at time $t \in [0, T]$ is given by $\partial_z F(t, x, u(x))$. Here $F : [0, T] \times \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is such that:

$$\text{for every } z \in \mathbb{R}^2 : \ (t, x) \to F(t, x, z) \text{ is } \mathcal{L}^1 \times \mathcal{L}^2 \text{ measurable on } [0, T] \times \Omega,$$

$$\text{for every } (t, x) \in [0, T] \times \Omega : \ z \to F(t, x, z) \text{ belongs to } C^1(\mathbb{R}^2),$$

and satisfies the following growth conditions

$$\begin{aligned} &a_0^F |z|^q - b_0^F(t, x) \le -F(t, x, z) \le a_1^F |z|^q + b_1^F(t, x), \\ &|\partial_z F(t, x, z)| \le a_2^F |z|^{q-1} + b_2^F(t, x) \end{aligned} \tag{3.8}$$

for every $(t, x, z) \in [0, T] \times \Omega \times \mathbb{R}^2$, with $a_0^F > 0$, $a_1^F > 0$ and $a_2^F > 0$, and where $b_0^F, b_1^F \in C^0([0, T]; L^1(\Omega))$, $b_2^F \in C^0([0, T]; L^{q'}(\Omega))$ are nonnegative functions, with $q' := q/(q-1)$.

In order to deal with time variations, we assume also that for every $(t,z) \in [0,T] \times \mathbb{R}^2$

$$F(t,x,z) = F(0,x,z) + \int_0^t \dot F(s,x,z)\,ds \qquad \text{for a.e. } x \in \Omega,$$

$$\partial_z F(t,x,z) = \partial_z F(0,x,z) + \int_0^t \partial_z \dot F(s,x,z)\,ds \qquad \text{for a.e. } x \in \Omega,$$

where $\dot F : [0,T] \times \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is such that

for all $z \in \mathbb{R}^2 :\ (t,x) \to \dot F(t,x,z)$ is $\mathcal{L}^1 \times \mathcal{L}^2$ measurable on $[0,T] \times \Omega$,

for all $(t,x) \in [0,T] \times \Omega :\ z \to \dot F(t,x,z)$ is of class C^1 on $\mathbb{R}^2$,

and satisfies the growth conditions

$$|\dot F(t,x,z)| \le a_3^F(t)|z|^{\dot q} + b_3^F(t,x),$$
$$|\partial_z \dot F(t,x,z)| \le a_4^F(t)|z|^{\dot q - 1} + b_4^F(t,x)$$

for all $(t,x,z) \in [0,T] \times \Omega \times \mathbb{R}^2$. Here $1 \le \dot q < q$, and $a_3^F, a_4^F \in L^1([0,T])$, $b_3^F \in L^1([0,T]; L^1(\Omega))$, $b_4^F \in L^1([0,T]; L^{\dot q'}(\Omega))$ are nonnegative functions with $\dot q' := \frac{\dot q}{\dot q - 1}$.

Under the previous assumptions, for every $t \in [0,T]$ the functionals

$$\mathcal{F}(t)(u) := \int_\Omega F(t,x,u(x))\,dx, \qquad \dot{\mathcal{F}}(t)(u) := \int_\Omega \dot F(t,x,u(x))\,dx \tag{3.9}$$

are well defined on $L^q(\Omega;\mathbb{R}^2)$ and $L^{\dot q}(\Omega;\mathbb{R}^2)$ respectively. Moreover we have that $\mathcal{F}(t)$ is of class C^1 on $L^q(\Omega;\mathbb{R}^2)$, with differential $\partial\mathcal{F}(t) : L^q(\Omega;\mathbb{R}^2) \to L^{q'}(\Omega;\mathbb{R}^2)$ defined by

$$\langle \partial\mathcal{F}(t)(u), v\rangle = \int_\Omega \partial_z F(t,x,u(x))v(x)\,dx, \qquad u,v \in L^q(\Omega;\mathbb{R}^2),$$

where $\langle\cdot,\cdot\rangle$ denotes now the duality pairing between $L^{q'}(\Omega;\mathbb{R}^2)$ and $L^q(\Omega;\mathbb{R}^2)$. $\dot{\mathcal{F}(t)}$ is C^1 on $L^{\dot q}(\Omega;\mathbb{R}^2)$ with differential defined by

$$\langle \partial\dot{\mathcal{F}}(t)(u), v\rangle = \int_\Omega \partial_z \dot F(t,x,u(x))v(x)\,dx, \qquad u,v \in L^{\dot q}(\Omega;\mathbb{R}^2),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $L^{\dot q'}(\Omega;\mathbb{R}^2)$ and $L^{\dot q}(\Omega;\mathbb{R}^2)$. For all $u, v \in L^q(\Omega;\mathbb{R}^2)$ and for all $t \in [0,T]$ we have

$$\mathcal{F}(t)(u) = \mathcal{F}(0)(u) + \int_0^t \dot{\mathcal{F}}(s)(u)\,ds,$$

$$\langle \partial\mathcal{F}(t)(u), v\rangle = \langle \partial\mathcal{F}(0)(u), v\rangle + \int_0^t \langle \partial\dot{\mathcal{F}}(s)(u), v\rangle\,ds. \tag{3.10}$$

Moreover we have that for every $t \in [0,T]$ and for every $u,\ v \in L^q(\Omega;\mathbb{R}^n)$

$$\alpha_0^{\mathcal{F}}\|u\|_q^q - \beta_0^{\mathcal{F}} \le -\mathcal{F}(t)(u) \le \alpha_1^{\mathcal{F}}\|u\|_q^q + \beta_1^{\mathcal{F}},$$
$$|\langle \partial\mathcal{F}(t)(u), v\rangle| \le (\alpha_2^{\mathcal{F}}\|u\|_q^{q-1} + \beta_2^{\mathcal{F}})\|v\|_q, \tag{3.11}$$
$$|\dot{\mathcal{F}}(t)(u)| \le \alpha_3^{\mathcal{F}}(t)\|u\|_{\dot q}^{\dot q} + \beta_3^{\mathcal{F}}(t), \tag{3.12}$$
$$|\langle \partial\dot{\mathcal{F}}(t)(u), v\rangle| \le (\alpha_4^{\mathcal{F}}(t)\|u\|_{\dot q}^{\dot q - 1} + \beta_4^{\mathcal{F}}(t))\|v\|_{\dot q}, \tag{3.13}$$

where $\alpha_0^{\mathcal{F}} > 0$, $\alpha_1^{\mathcal{F}} > 0$, $\alpha_2^{\mathcal{F}} > 0$, $\beta_0^{\mathcal{F}} \ge 0$, $\beta_1^{\mathcal{F}} \ge 0$, $\beta_2^{\mathcal{F}} \ge 0$ are positive constants, and $\alpha_3^{\mathcal{F}}, \alpha_4^{\mathcal{F}}, \beta_3^{\mathcal{F}}, \beta_4^{\mathcal{F}} \in L^1([0,T])$ are nonnegative functions.

The surface forces. The density of the surface forces on $\partial_S\Omega$ at time t under the deformation u is given by $\partial_z G(t,x,u(x))$, where $G : [0,T] \times \partial_S\Omega \times \mathbb{R}^2 \to \mathbb{R}$ is such that

for every $z \in \mathbb{R}^2 :\ (t,x) \to G(t,x,z)$ is $\mathcal{L}^1 \times \mathcal{H}^1$-measurable,

for every $(t,x) \in [0,T] \times \partial_S\Omega :\ z \to G(t,x,z)$ belongs to $C^1(\mathbb{R}^2)$,

and satisfies the growth conditions

$$
\begin{aligned}
&- a_0^G(t,x)|z| - b_0^G(t,x) \le -G(t,x,z) \le a_1^G|z|^r + b_1^G(t,x),\\
&|\partial_z G(t,x,z)| \le a_2^G|z|^{r-1} + b_2^G(t,x),
\end{aligned}
$$

for every $(t,x,z) \in [0,T] \times \partial_S\Omega \times \mathbb{R}^2$. Here r is an exponent related to the trace operators on Sobolev spaces: if $p < 2$, then we suppose that $p \le r \le \frac{p}{2-p}$, while if $p \ge 2$, we suppose $p \le r$. Moreover $a_1^G \ge 0$, $a_2^G \ge 0$ are two nonnegative constants, and $a_0^G \in L^\infty([0,T];L^{r'}(\partial_S\Omega))$, $b_0^G, b_1^G \in C^0([0,T];L^1(\partial_S\Omega))$, and $b_2^G \in C^0([0,T];L^{r'}(\partial_S\Omega))$ are nonnegative functions with $r' := r/(r-1)$

We assume that for every $(t,z) \in [0,T] \times \mathbb{R}^2$

$$
\begin{aligned}
&G(t,x,z) = G(0,x,z) + \int_0^t \dot G(s,x,z)ds \qquad \text{for } \mathcal{H}^1\text{-a.e. } x \in \partial_S\Omega,\\
&\partial_z G(t,x,z) = \partial_z G(0,x,z) + \int_0^t \partial_z \dot G(s,x,z)ds \qquad \text{for } \mathcal{H}^1\text{-a.e. } x \in \partial_S\Omega,
\end{aligned}
$$

where $\dot G : [0,T] \times \partial_S\Omega \times \mathbb{R}^2 \to \mathbb{R}$ is such that

$$
\begin{aligned}
&\text{for all } z \in \mathbb{R}^2 : \ (t,x) \to \dot G(t,x,z) \text{ is } \mathcal{L}^1 \times \mathcal{H}^1\text{-measurable},\\
&\text{for all } (t,x) \in [0,T] \times \partial_S\Omega : \ z \to \dot G(t,x,z) \text{ belongs to } C^1(\mathbb{R}^2),
\end{aligned}
$$

and satisfies the the growth conditions

$$
\begin{aligned}
&|\dot G(t,x,z)| \le a_3^G(t)|z|^r + b_3^G(t,x),\\
&|\partial_z \dot G(t,x,z)| \le a_4^G(t)|z|^{r-1} + b_4^G(t,x)
\end{aligned}
$$

for all $(t,x,z) \in [0,T] \times \partial_S\Omega \times \mathbb{R}^2$. Here a_3^G, $a_4^G \in L^1([0,T])$, $b_3^G \in L^1([0,T];L^1(\partial_S\Omega))$ and $b_4^G \in L^1([0,T];L^{r'}(\partial_S\Omega))$ are nonnegative functions.

By the previous assumptions, the following functionals on $L^r(\partial_S\Omega;\mathbb{R}^2)$

$$
\mathcal{G}(t)(u) := \int_{\partial_S\Omega} G(t,x,u(x))\, d\mathcal{H}^1(x), \quad \dot{\mathcal{G}}(t)(u) := \int_{\partial_S\Omega} \dot G(t,x,u(x))\, d\mathcal{H}^1(x) \tag{3.14}
$$

are well defined. For every $t \in [0,T]$ we have that $\mathcal{G}(t)$ is of class C^1 on $L^r(\partial_S\Omega;\mathbb{R}^2)$ and its differential is given by

$$
\langle \partial\mathcal{G}(t)(u), v\rangle = \int_{\partial_S\Omega} \partial_z G(t,x,u(x))v(x)\, d\mathcal{H}^1(x), \qquad u,v \in L^r(\partial_S\Omega;\mathbb{R}^2),
$$

where $\langle\cdot,\cdot\rangle$ denotes now the duality pairing between $L^{r'}(\partial_S\Omega;\mathbb{R}^2)$ and $L^r(\partial_S\Omega;\mathbb{R}^2)$. Moreover, $\dot{\mathcal{G}}(t)$ is of class C^1 on $L^r(\partial_S\Omega;\mathbb{R}^2)$, and its differential is given by

$$
\langle \partial\dot{\mathcal{G}}(t)(u), v\rangle = \int_{\partial_S\Omega} \partial_z \dot G(t,x,u(x))v(x)\, d\mathcal{H}^1(x)
$$

for all $u,v \in L^r(\partial_S\Omega;\mathbb{R}^2)$. Finally we have

$$
\mathcal{G}(t)(u) = \mathcal{G}(0)(u) + \int_0^t \dot{\mathcal{G}}(s)(u)\, ds,
$$

$$
\langle \partial\mathcal{G}(t)(u), v\rangle = \langle \partial\mathcal{G}(0)(u), v\rangle + \int_0^t \langle \partial\dot{\mathcal{G}}(s)(u), v\rangle\, ds,
$$

for every $u,v \in L^r(\partial_S\Omega;\mathbb{R}^2)$.

Let $\Omega_S \subseteq \Omega \setminus \overline{\Omega}_B$ be open with Lipschitz boundary, and such that $\partial_S\Omega \subseteq \partial\Omega_S$; the trace operator from $W^{1,p}(\Omega_S;\mathbb{R}^2)$ into $L^r(\partial\Omega_S;\mathbb{R}^2)$ is then compact, and so there exists a constant $\gamma_S > 0$ such that

$$
\|u\|_{r,\partial_S\Omega} \le \gamma_S(\|\nabla u\|_{p,\Omega_S} + \|u\|_{p,\Omega_S}) \tag{3.15}
$$

for every $u \in W^{1,p}(\Omega_S;\mathbb{R}^2)$. By the previous assumptions, we have that there exist six nonnegative constants $\alpha_0^{\mathcal{G}}$, $\alpha_1^{\mathcal{G}}$, $\alpha_2^{\mathcal{G}}$, $\beta_0^{\mathcal{G}}$, $\beta_1^{\mathcal{G}}$, $\beta_2^{\mathcal{G}}$ and four nonnegative functions $\alpha_3^{\mathcal{G}}$, $\alpha_4^{\mathcal{G}}$, $\beta_3^{\mathcal{G}}$, $\beta_4^{\mathcal{G}} \in L^1([0,T])$, such that

$$-\alpha_0^G \|u\|_{r,\partial_S\Omega} - \beta_0^G \le -\mathcal{G}(t)(u) \le \alpha_1^{\mathcal{G}} \|u\|^r_{r,\partial_S\Omega} + \beta_1^G,$$

$$|\langle \partial\mathcal{G}(t)(u), v\rangle| \le (\alpha_2^{\mathcal{G}} \|u\|^{r-1}_{r,\partial_S\Omega} + \beta_2^{\mathcal{G}})\|v\|_{r,\partial_S\Omega}, \tag{3.16}$$

$$|\dot{\mathcal{G}}(t)(u)| \le \alpha_3^{\mathcal{G}}(t)\|u\|^r_{r,\partial_S\Omega} + \beta_3^{\mathcal{G}}(t), \tag{3.17}$$

$$|\langle \partial\dot{\mathcal{G}}(t)(u), v\rangle| \le (\alpha_4^{\mathcal{G}}(t)\|u\|^{r-1}_{r,\partial_S\Omega} + \beta_4^{\mathcal{G}}(t))\|v\|_{r,\partial_S\Omega}$$

for every $t \in [0,T]$ and $u, v \in L^r(\partial_S\Omega;\mathbb{R}^2)$.

Configurations with finite energy. The deformations on the boundary $\partial_D\Omega$ are given by (the traces of) functions $g \in W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)$, where p, q are the exponents in (3.5) and (3.8) respectively. Given a crack $\Gamma \in \mathcal{R}(\overline{\Omega}_B;\partial_N\Omega)$ and a boundary deformation g, the set of *admissible deformations with finite energy* relative to (g,Γ) is defined by

$$AD(g,\Gamma) := \{u \in GSBV^p_q(\Omega;\mathbb{R}^2) : S(u) \,\tilde{\subset}\, \Gamma, u = g \ \mathcal{H}^1\text{-a.e. on } \partial_D\Omega \setminus \Gamma\},$$

where we recall that

$$GSBV^p_q(\Omega;\mathbb{R}^2) := GSBV^p(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2),$$

and the equality $u = g$ on $\partial_D\Omega \setminus \Gamma$ is intended in the sense of traces (see [13, Section 2]).

Note that if $u \in GSBV^p_q(\Omega;\mathbb{R}^2)$, then $\mathcal{W}(u) < +\infty$ and $|\mathcal{F}(t)(u)| < +\infty$ for all $t \in [0,T]$. Moreover since $\Gamma \in \mathcal{R}(\overline{\Omega}_B;\partial_N\Omega)$ and $S(u) \,\tilde{\subset}\, \Gamma$, we have that $u \in W^{1,p}(\Omega_S;\mathbb{R}^2) \cap L^q(\Omega_S;\mathbb{R}^2)$ so that $\mathcal{G}(t)(u)$ is well defined and $|\mathcal{G}(t)(u)| < +\infty$ for all $t \in [0,T]$. Notice that there exists always a deformation without crack which satisfies the boundary condition, namely the function g itself.

The total energy. For every $t \in [0,T]$, the total energy relative to the configuration (u,Γ) with $u \in AD(g,\Gamma)$ is given by

$$\mathcal{E}(t)(u,\Gamma) := \mathcal{E}^{el}(t)(u) + \mathcal{E}^s(\Gamma), \tag{3.18}$$

where

$$\mathcal{E}^{el}(t)(u) := \mathcal{W}(u) - \mathcal{F}(t)(u) - \mathcal{G}(t)(u), \tag{3.19}$$

and $\mathcal{W}$, $\mathcal{F}(t)$, $\mathcal{G}(t)$ and $\mathcal{E}^s$ are defined in (3.3), (3.9), (3.14) and (3.1) respectively. It turns out that there exist four constants $\alpha_0^{\mathcal{E}} > 0$, $\alpha_1^{\mathcal{E}} > 0$, $\beta_0^{\mathcal{E}} \ge 0$, $\beta_1^{\mathcal{E}} \ge 0$ such that

$$\mathcal{E}^{el}(t)(u) \ge \alpha_0^{\mathcal{E}}(\|\nabla u\|_p^p + \|u\|_q^q) - \beta_0^{\mathcal{E}}, \tag{3.20}$$

$$\mathcal{E}^{el}(t)(u) \le \alpha_1^{\mathcal{E}}(\|\nabla u\|_p^p + \|u\|_q^q + \|u\|^r_{r,\partial_S\Omega}) + \beta_1^{\mathcal{E}},$$

for every $t \in [0,T]$ and $u \in GSBV^p_q(\Omega;\mathbb{R}^2)$.

The time dependent boundary deformations. We will consider boundary deformations $g(t)$ such that

$$t \to g(t) \in AC([0,T]; W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)),$$

so that

$$t \to \dot{g}(t) \in L^1([0,T]; W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)),$$

and

$$t \to \nabla\dot{g}(t) \in L^1([0,T]; L^p(\Omega; M^{2\times 2})).$$

The existence result. Let $\Gamma_0 \in \mathcal{R}(\overline{\Omega}_B;\partial_N\Omega)$ be a preexisting crack. The next Theorem proved in [13] establishes the existence of a quasistatic evolution with preexisting crack Γ_0.

Theorem 3.1. *Let $\Gamma_0 \in \mathcal{R}(\overline{\Omega}_B;\partial_N\Omega)$ be a preexisting crack. Then there exists a quasistatic evolution with preexisting crack Γ_0 and boundary deformation $g(t)$, i.e., there exists a function $t \to (u(t),\Gamma(t))$ from $[0,T]$ to $GSBV^p_q(\Omega;\mathbb{R}^2) \times \mathcal{R}(\overline{\Omega}_B;\partial_N\Omega)$ with the following properties:*

(a) $(u(0), \Gamma(0))$ *is such that*

$$\mathcal{E}(0)(u(0), \Gamma(0)) = \min\{\mathcal{E}(0)(v, \Gamma) : v \in AD(g(0), \Gamma), \Gamma_0 \,\tilde{\subset}\, \Gamma\};$$

(b) $u(t) \in AD(g(t), \Gamma(t))$ *for all* $t \in [0, T]$*;*

(c) *irreversibility:* $\Gamma_0 \,\tilde{\subset}\, \Gamma(s) \tilde{\subset} \Gamma(t)$ *whenever* $0 \le s < t \le T$*;*

(d) *static equilibrium: for all* $t \in [0, T]$

$$\mathcal{E}(t)(u(t), \Gamma(t)) = \min\{\mathcal{E}(t)(v, \Gamma) : v \in AD(g(t), \Gamma), \Gamma(t) \,\tilde{\subset}\, \Gamma\};$$

(e) *nondissipativity: the function* $t \to E(t) := \mathcal{E}(t)(u(t), \Gamma(t))$ *is absolutely continuous on* $[0, T]$*, and for a.e.* $t \in [0, T]$

$$\dot{E}(t) = \langle \partial \mathcal{W}(\nabla u(t)), \nabla \dot{g}(t) \rangle - \langle \partial \mathcal{F}(t)(u(t)), \dot{g}(t) \rangle - \dot{\mathcal{F}}(t)(u(t)) \\ - \langle \partial \mathcal{G}(t)(u(t)), \dot{g}(t) \rangle - \dot{\mathcal{G}}(t)(u(t)). \tag{3.21}$$

The next theorem gives a compactness and lower semicontinuity result with respect to weak convergence in $GSBV^p_q(\Omega, \mathbb{R}^2)$ which will be often used in the next sections.

Theorem 3.2. *Let* $t_k \in [0, T]$ *with* $t_k \to t$*, and let* $(u_k) \subset GSBV^p_q(\Omega; \mathbb{R}^2)$*,* $C \in]0; +\infty[$ *such that* $S(u_k) \,\tilde{\subset}\, \overline{\Omega}_B$ *and*

$$\mathcal{E}^{el}(t_k)(u_k) + \mathcal{E}^s(S(u_k)) \le C,$$

where $\mathcal{E}^{el}$ *and* $\mathcal{E}^s$ *are defined as in* (3.19) *and* (3.1)*. Then there exists a subsequence* $(u_{k_h})_{h \in \mathbb{N}}$ *converging to some* u *weakly in* $GSBV^p_q(\Omega; \mathbb{R}^2)$ *such that* $S(u) \,\tilde{\subset}\, \overline{\Omega}_B$*,*

$$\mathcal{E}^{el}(t)(u) \le \liminf_{h \to \infty} \mathcal{E}^{el}(t_{k_h})(u_{k_h}),$$

and

$$\mathcal{E}^s(S(u)) \le \liminf_{h \to \infty} \mathcal{E}^s(S(u_{k_h})).$$

Proof. By (3.20) and (3.2), we have that there exists $C' \in]0, +\infty[$ such that

$$\|\nabla u_k\|_p^p + \|u_k\|_q^q + \mathcal{H}^1(S(u_k)) \le C'.$$

Then we can apply Theorem 2.1 with $g(x, u_k) = |u_k|^q$, obtaining a subsequence $(u_{k_h})_{h \in \mathbb{N}}$ and $u \in GSBV^p(\Omega; \mathbb{R}^2)$ such that (2.1) holds: in particular we may assume that $u_{k_h} \to u$ pointwise a.e.. We have $u_{k_h} \to u$ strongly in $L^1(\Omega; \mathbb{R}^2)$, and by Fatou's Lemma we have that $u \in L^q(\Omega; \mathbb{R}^2)$ so that $u \in GSBV^p_q(\Omega; \mathbb{R}^2)$. We conclude $u_{k_h} \rightharpoonup u$ weakly in $GSBV^p_q(\Omega; \mathbb{R}^2)$. By [2, Theorem 3.7] we have that

$$\mathcal{E}^s(S(u)) \le \liminf_h \mathcal{E}^s(S(u_{k_h})),$$

by [18] we have that

$$\int_\Omega W(x, \nabla u)\, dx \le \liminf_h \int_\Omega W(x, \nabla u_{k_h})\, dx,$$

and by Fatou's Lemma (in the limsup version) we have

$$\limsup_h \int_\Omega F(t_{k_h}, x, u_{k_h}(x))\, dx \le \int_\Omega F(t, x, u(x))\, dx.$$

Since $(u_{h_k})_{|\Omega_S}$ is bounded in $W^{1,p}(\Omega_S; \mathbb{R}^2) \cap L^q(\Omega_S; \mathbb{R}^2)$, and the trace operator from $W^{1,p}(\Omega_S; \mathbb{R}^2)$ into $L^r(\Omega_S; \mathbb{R}^2)$ is compact, we get

$$\lim_h \mathcal{G}(t_{k_h})(u_{k_h}) = \mathcal{G}(t)(u),$$

and so the proof is thus concluded. □

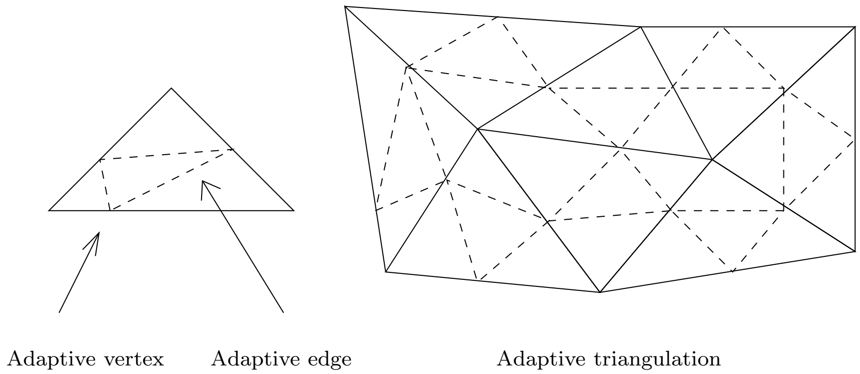

FIGURE 1.

4. The finite element space and the transfer of jump

Let $\Omega \subseteq \mathbb{R}^2$ be a polygonal set and let us fix two positive constants $0 < c_1 < c_2 < +\infty$. By a *regular triangulation* of Ω of size ε we intend a finite family of (closed) triangles T_i such that $\overline{\Omega} = \bigcup_i T_i$, $T_i \cap T_j$ is either empty or equal to a common edge or to a common vertex, and each T_i contains a ball of diameter $c_1\varepsilon$ and is contained in a ball of diameter $c_2\varepsilon$.

We indicate by $\mathcal{R}_\varepsilon(\Omega)$ the family of all regular triangulations of Ω of size ε. It turns out that there exist $0 < \vartheta_1 < \vartheta_2 < \pi$ such that for all triangle T belonging to a regular triangulation $\mathbf{T} \in \mathcal{R}_\varepsilon(\Omega)$, the inner angles of T are between ϑ_1 and ϑ_2. Moreover, every edge of T has length greater than $c_1\varepsilon$ and lower than $c_2\varepsilon$.

Let $\varepsilon > 0$, $\mathbf{R}_\varepsilon \in \mathcal{R}_\varepsilon(\Omega)$, and let $a \in]0, \frac{1}{2}[$. Let us consider a triangulation $\mathbf{T}$ nested in $\mathbf{R}_\varepsilon$ obtained dividing each triangle $T \in \mathbf{R}_\varepsilon$ into four triangles taking over every edge $[x, y]$ of T a knot z which satisfies

$$z = tx + (1-t)y, \qquad t \in [a, 1-a].$$

We will call these vertices *adaptive*, the triangles obtained gluing these points *adaptive triangles*, and their edges *adaptive edges*. We denote by $\mathcal{T}_{\varepsilon,a}(\Omega)$ the set of triangulations $\mathbf{T}$ constructed in this way. Note that for all $\mathbf{T} \in \mathcal{T}_{\varepsilon,a}(\Omega)$ there exist $0 < c_1^a < c_2^a < +\infty$ such that every $T_i \in \mathbf{T}$ contains a ball of diameter $c_1^a\varepsilon$ and is contained in a ball of diameter $c_2^a\varepsilon$. Then there exist $0 < \vartheta_1^a < \vartheta_2^a < \pi$ such that for all T belonging to $\mathbf{T} \in \mathcal{T}_{\varepsilon,a}(\Omega)$, the inner angles of T are between ϑ_1^a and ϑ_2^a. Moreover, every edge of T has length greater than $c_1^a\varepsilon$ and lower than $c_2^a\varepsilon$.

From now on for all $\varepsilon > 0$ we fix $\mathbf{R}_\varepsilon \in \mathcal{R}_\varepsilon(\Omega)$. We suppose that the brittle part Ω_B and the region Ω_S introduced before for the model of quasistatic growth of fractures are composed of triangles of $\mathbf{R}_\varepsilon$ for all ε. Moreover we suppose that $\partial_D\Omega$ and $\partial_S\Omega$ are composed of edges of $\mathbf{R}_\varepsilon$ for all ε up to a finite number of points. Finally, in order to deal with the deformation at the boundary, it will be useful to consider Ω_D polygonal such that $\Omega_D \cap \Omega = \emptyset$, and $\partial\Omega_D \cap \partial\Omega = \partial_D\Omega$ up to a finite number of points. We set

$$\Omega' := \Omega \cup \Omega_D \cup \partial_D\Omega, \tag{4.1}$$

and we suppose that the regular triangulation $\mathbf{R}_\varepsilon$ can be extended to a regular triangulation of Ω', so that every triangulation $\mathbf{T}$ in $\mathcal{T}_{\varepsilon,a}(\Omega)$ can be extended to a triangulation of $\mathcal{T}_{\varepsilon,a}(\Omega')$ considering the middle points of the new edges as adaptive vertices: we indicate these extended triangulation with the same symbol $\mathbf{T}$.

We consider the following discontinuous finite element space. We indicate by $\mathcal{A}_{\varepsilon,a}(\Omega; \mathbb{R}^2)$ the set of all $u : \Omega \to \mathbb{R}^2$ such that there exists a triangulation $\mathbf{T}(u) \in \mathcal{T}_{\varepsilon,a}(\Omega)$ nested in $\mathbf{R}_\varepsilon$ with u affine on every triangle $T \in \mathbf{T}(u)$. For every $u \in \mathcal{A}_{\varepsilon,a}(\Omega; \mathbb{R}^2)$, we indicate by $S(u)$ the family of edges of $\mathbf{T}(u)$ inside Ω across which u is discontinuous. Notice that $u \in SBV(\Omega; \mathbb{R}^2)$ and that the notation is consistent with the usual one employed in the theory of functions of bounded variation. Let us set

$$\mathcal{AF}_\varepsilon(\Omega; \mathbb{R}^2) := \{u : \Omega \to \mathbb{R}^2 \,:\, u \text{ is continuous and affine on each triangle } T \in \mathbf{R}_\varepsilon\}. \tag{4.2}$$

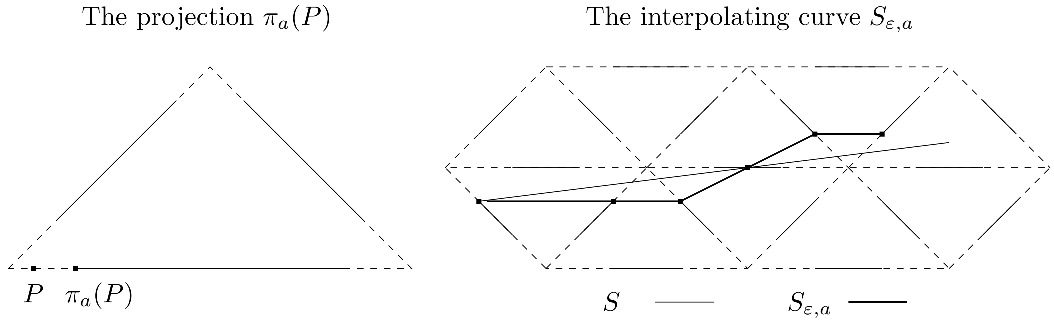

Figure 2.

The discretization of the problem will be carried out using the space

$$\mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2) := \{u \in \mathcal{A}_{\varepsilon,a}(\Omega;\mathbb{R}^2)\,:\, S(u) \subseteq \overline{\Omega}_B\}. \tag{4.3}$$

Given any $g \in \mathcal{AF}_\varepsilon(\Omega;\mathbb{R}^2)$, for every $u \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$ let

$$S^g_D(u) := \{x \in \partial_D\Omega \,:\, u(x) \neq g(x)\}, \tag{4.4}$$

that is $S^g_D(u)$ denotes the set of edges of $\partial_D\Omega$ at which the boundary condition is not satisfied. For every $u \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$, let us also set

$$S^g(u) := S(u) \cup S^g_D(u). \tag{4.5}$$

An essential tool in the approximation result of this paper is Proposition 4.2 which generalizes the piecewise affine transfer of jump [17, Proposition 5.1] to the case of vector valued functions with bulk energy $\mathcal{E}^{el}$ and surface energy $\mathcal{E}^s$ of the form (3.19) and (3.1) respectively.

In order to deal with the surface energy $\mathcal{E}^s$ we will need the following geometric construction. Let $S \subseteq \Omega$ be a segment and let us suppose that S intersects the edges of $\mathbf{R}_\varepsilon$ at most in one point for all $\varepsilon > 0$. Let $a \in]0, \frac{1}{2}[$, and let $P = S \cap \zeta$, where $\zeta = [x, y]$ is an edge of $\mathbf{R}_\varepsilon$: we indicate with $\pi_a(P)$ the projection of P on the segment $\{tx + (1-t)y \,:\, t \in [a, 1-a]\}$. The *interpolating curve* $S_{\varepsilon,a}$ of S in $\mathbf{R}_\varepsilon$ with parameter a is given connecting all the $\pi_a(P)$'s belonging to the same triangle of $\mathbf{R}_\varepsilon$ (see Figure 2).

Lemma 4.1. *Under the previous assumptions, there exists a function $\eta(a)$ independent of S with $\eta(a) \to 0$ as $a \to 0$ such that*

$$\limsup_{\varepsilon\to 0} |\mathcal{E}^s(S_{\varepsilon,a}) - \mathcal{E}^s(S)| \leq \eta(a)\mathcal{E}^s(S),$$

where $\mathcal{E}^s$ is defined in (3.1).

Proof. By (3.2), we have that there exist ω and $K_3 > 0$ such that for all $x_1, x_2 \in \overline{\Omega}$ and $|\nu_1| = |\nu_2| = 1$

$$|k(x_1,\nu_1) - k(x_2,\nu_2)| \leq \omega(|x_1 - x_2|) + K_3|\nu_1 - \nu_2|,$$

where $\omega\,:]0,+\infty[\to]0,+\infty[$ is a decreasing function such that $\omega(s) \to 0$ as $s \to 0$. Let $T \in \mathbf{R}_\varepsilon$ be such that $T \cap S \neq \emptyset$, and let us choose $x_T \in T \cap S$ and $x^{\varepsilon,a}_T \in T \cap S_{\varepsilon,a}$. Let $c_2 > 0$ denote the characteristic constant of $\mathbf{R}_\varepsilon$ such that every $T \in \mathbf{R}_\varepsilon$ is contained in a ball of diameter $c_2\varepsilon$. Then we have

$$\begin{aligned}
&\left|\int_{S_{\varepsilon,a}\cap T} k(x,\nu^{\varepsilon,a}_T)\,d\mathcal{H}^1 - \int_{S\cap T} k(x,\nu_T)\,d\mathcal{H}^1\right| \\
&\leq \left|\int_{S_{\varepsilon,a}\cap T} k(x^{\varepsilon,a}_T,\nu^{\varepsilon,a}_T)\,d\mathcal{H}^1 - \int_{S\cap T} k(x_T,\nu_T)\,d\mathcal{H}^1\right| + \omega(c_2\varepsilon)\mathcal{H}^1(S_{\varepsilon,a}\cap T) + \omega(c_2\varepsilon)\mathcal{H}^1(S\cap T) \\
&\quad\leq \left|k(x^{\varepsilon,a}_T,\nu^{\varepsilon,a}_T)\mathcal{H}^1(S_{\varepsilon,a}\cap T) - k(x_T,\nu_T)\mathcal{H}^1(S\cap T)\right| + \omega(c_2\varepsilon)\left[\mathcal{H}^1(S_{\varepsilon,a}\cap T) + \mathcal{H}^1(S\cap T)\right],
\end{aligned}$$

where $\nu_T^{\varepsilon,a}, \nu_T$ are the (constant) normal to $S_{\varepsilon,a} \cap T$ and $S \cap T$ respectively. We have

$$\begin{aligned}
&\left|k(x_T^{\varepsilon,a}, \nu_T^{\varepsilon,a})\mathcal{H}^1(S_{\varepsilon,a} \cap T) - k(x_T, \nu_T)\mathcal{H}^1(S \cap T)\right| \\
&\quad \le k(x_T^{\varepsilon,a}, \nu_T^{\varepsilon,a}) \left|\mathcal{H}^1(S_{\varepsilon,a} \cap T) - \mathcal{H}^1(S \cap T)\right| + \left|k(x_T^{\varepsilon,a}, \nu_T^{\varepsilon,a}) - k(x_T, \nu_T)\right| \mathcal{H}^1(S \cap T) \\
&\quad \le K_2 \left|\mathcal{H}^1(S_{\varepsilon,a} \cap T) - \mathcal{H}^1(S \cap T)\right| + \omega(|x_T^{\varepsilon,a} - x_T|)\mathcal{H}^1(S \cap T) + K_3|\nu_T^{\varepsilon,a} - \nu_T|\mathcal{H}^1(S \cap T),
\end{aligned}$$

where K_2 is defined in (3.2). We are now ready to conclude: in fact, following [19, Lemma 5.2.2], we can choose the orientation of $\nu_T^{\varepsilon,a}$ in such a way that

$$|\nu_T^{\varepsilon,a} - \nu_T|\mathcal{H}^1(S \cap T) \le D_2 a\varepsilon, \qquad \left|\mathcal{H}^1(S_{\varepsilon,a} \cap T) - \mathcal{H}^1(S \cap T)\right| \le D_1 a\varepsilon,$$

with $D_1, D_2 > 0$ independent of T, ε, a. Then, summing up the preceding inequalities, recalling that the number of triangles of $\mathbf{R}_\varepsilon$ intersecting S is less than $\tilde{c}\varepsilon^{-1}\mathcal{H}^1(S)$ for ε small enough, with $\tilde{c}$ independent of S and ε (see for example [17, Lemma 2.5]), we obtain

$$\limsup_{\varepsilon \to 0} |\mathcal{E}^s(S_{\varepsilon,a}) - \mathcal{E}^s(S)| \le \rho(a)\mathcal{H}^1(S),$$

where $\rho(a) := \tilde{c}(K_2 D_1 + K_3 D_2)a$. In view of (3.2), we conclude that

$$\limsup_{\varepsilon \to 0} |\mathcal{E}^s(S_{\varepsilon,a}) - \mathcal{E}^s(S)| \le K_1^{-1}\rho(a)\mathcal{E}^s(S),$$

and so the proof is concluded choosing $\eta(a) := K_1^{-1}\rho(a)$. □

For all $u \in GSBV_q^p(\Omega; \mathbb{R}^2)$ and for all $g \in W^{1,p}(\Omega; \mathbb{R}^2) \cap L^q(\Omega; \mathbb{R}^2)$, let us set

$$S^g(u) := S(u) \cup \{x \in \partial_D \Omega \,:\, u(x) \neq g(x)\}, \tag{4.6}$$

where the inequality is intended in the sense of traces. We are now in a position to state the piecewise affine transfer of jump proposition in our setting.

Proposition 4.2. *Let $a \in]0, \frac{1}{2}[$, and for all $i = 1, \ldots, m$ let*

$$u_\varepsilon^i \in \mathcal{A}_{\varepsilon,a}^B(\Omega; \mathbb{R}^2), \qquad u^i \in GSBV_q^p(\Omega; \mathbb{R}^2)$$

be such that

$$u_\varepsilon^i \rightharpoonup u^i \qquad \textit{weakly in } GSBV_q^p(\Omega; \mathbb{R}^2).$$

Let moreover $g_\varepsilon^i, h_\varepsilon \in \mathcal{AF}_\varepsilon(\Omega; \mathbb{R}^2)$, $g^i, h \in W^{1,p}(\Omega; \mathbb{R}^2) \cap L^q(\Omega; \mathbb{R}^2)$ be such that

$$g_\varepsilon^i \to g^i, \quad h_\varepsilon \to h \qquad \textit{strongly in } W^{1,p}(\Omega; \mathbb{R}^2) \cap L^q(\Omega; \mathbb{R}^2).$$

Then for every $v \in GSBV_q^p(\Omega; \mathbb{R}^2)$ with $S(v) \,\tilde{\subset}\, \overline{\Omega}_B$, there exists $v_\varepsilon \in \mathcal{A}_{\varepsilon,a}^B(\Omega; \mathbb{R}^2)$ such that

$$\begin{aligned}
\nabla v_\varepsilon \to \nabla v \qquad &\textit{strongly in } L^p(\Omega; M^{2\times 2}), \\
v_\varepsilon \to v \qquad &\textit{strongly in } L^q(\Omega; \mathbb{R}^2),
\end{aligned}$$

and such that

$$\limsup_{\varepsilon \to 0} \mathcal{E}^s\left(S^{h_\varepsilon}(v_\varepsilon) \setminus \bigcup_{i=1}^m S^{g_\varepsilon^i}(u_\varepsilon^i)\right) \le \mu(a)\mathcal{E}^s\left(S^h(v) \setminus \bigcup_{i=1}^m S^{g^i}(u^i)\right),$$

where $\mu(a)$ depends only on a, $\mu(a) \to 1$ as $a \to 0$, and $\mathcal{E}^s$ is defined in (3.1)*. In particular for all $t \in [0, T]$ and for all $t_\varepsilon \to t$ we have*

$$\mathcal{E}^{el}(t_\varepsilon)(v_\varepsilon) \to \mathcal{E}^{el}(t)(v),$$

where $\mathcal{E}^{el}$ is defined in (3.19)*.*

The proof of Proposition 4.2 can be obtained from that of [17, Proposition 5.1] taking into account the following modifications. We can consider v scalar valued since vector valued maps can be easily dealt componentwise. Even if the surface energy is of the form (3.1), we can still restrict ourselves to the case in which v has piecewise linear jumps outside a suitable neighborhood of $\bigcup_{i=1}^m S^{g^i}(u^i)$ by using the density result of [10]. In order to approximate the piecewise linear jumps, we use Lemma 4.1. Finally the fact that $p \neq 2$ prevents us from considering the piecewise jumps as union of disjoint segments: we overcome this difficulty choosing $v_\varepsilon = 0$ in the regular triangles which contain the intersection points, and then interpolating v outside as in [17, Proposition 5.1].

5. Preexisting cracks and their approximation

In Section 7, we will need to approximate the surface energy of a given preexisting crack Γ^0. We take the initial crack in the class

$$\boldsymbol{\Gamma}(\Omega) := \{\Gamma \tilde{\subset} \overline{\Omega}_B \,:\, \mathcal{H}^1(\Gamma) < +\infty,\, \Gamma = S^h(z) \text{ for some } h \in W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2) \text{ and } z \in GSBV^p_q(\Omega;\mathbb{R}^2)\}. \tag{5.1}$$

Notice that it is not restrictive to assume $h \equiv 0$. We take as discretization of $\boldsymbol{\Gamma}(\Omega)$ the following class

$$\boldsymbol{\Gamma}_{\varepsilon,a}(\Omega) := \{\Gamma \tilde{\subset} \overline{\Omega}_B \,:\, \mathcal{H}^1(\Gamma) < +\infty,\, \Gamma = S^0(z) \text{ for some } z \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)\}. \tag{5.2}$$

We have the following approximation result.

Proposition 5.1. *Let $\Gamma^0 \in \boldsymbol{\Gamma}(\Omega)$. Then for every $\varepsilon > 0$ and $a \in]0, \frac{1}{2}[$ there exists $\Gamma^0_{\varepsilon,a} \in \boldsymbol{\Gamma}_{\varepsilon,a}(\Omega)$ such that*

$$\lim_{\varepsilon,a\to 0} \mathcal{E}^s(\Gamma^0_{\varepsilon,a}) = \mathcal{E}^s(\Gamma^0),$$

where $\mathcal{E}^s$ is defined in (3.1).

Moreover let $g_\varepsilon \in \mathcal{AF}_\varepsilon(\Omega;\mathbb{R}^2)$, $g \in W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)$ be such that as $\varepsilon \to 0$

$$g_\varepsilon \to g \text{ strongly in } W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2),$$

and let us consider

$$F_{\varepsilon,a}(v) := \begin{cases} \mathcal{E}^{el}(0)(v) + \mathcal{E}^s\left(S^{g_\varepsilon}(v) \setminus \Gamma^0_{\varepsilon,a}\right) & \text{if } v \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2), \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{R}^2), \end{cases}$$

and

$$F(v) := \begin{cases} \mathcal{E}^{el}(0)(v) + \mathcal{E}^s\left(S^{g}(v) \setminus \Gamma^0\right) & \text{if } v \in GSBV^p_q(\Omega;\mathbb{R}^2), S(v) \tilde{\subset} \overline{\Omega}_B, \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{R}^2), \end{cases}$$

where $\mathcal{E}^{el}$ is defined in (3.19). *Then the family $(F_{\varepsilon,a})$ Γ-converges to F in the strong topology of $L^1(\Omega;\mathbb{R}^2)$ as $\varepsilon \to 0$ and $a \to 0$.*

Proof. Let us consider $\Gamma^0 \in \boldsymbol{\Gamma}(\Omega)$ with $\Gamma^0 = S^0(z)$ for some $z \in GSBV^p_q(\Omega;\mathbb{R}^2)$. Then by Proposition 4.2 for every $\varepsilon > 0$ and $a \in (0, \frac{1}{2})$, there exists $\tilde{z}_{\varepsilon,a} \in \mathcal{A}_{\varepsilon,a}(\Omega;\mathbb{R}^2)$ such that for $\varepsilon \to 0$ and for all a

$$\nabla \tilde{z}_{\varepsilon,a} \to \nabla z \qquad \text{strongly in } L^p(\Omega; M^{2\times 2}),$$
$$\tilde{z}_{\varepsilon,a} \to z \qquad \text{strongly in } L^q(\Omega;\mathbb{R}^2),$$

and

$$\limsup_{\varepsilon\to 0} \mathcal{E}^s(S^0(\tilde{z}_{\varepsilon,a})) \leq \mu(a)\mathcal{E}^s(S^0(z))$$

with $\mu(a) \to 1$ as $a \to 0$, where $\mathcal{E}^s$ is defined in (3.1). Let $a_i \searrow 0$, and let $\varepsilon_i \searrow 0$ be such that for all $\varepsilon \leq \varepsilon_i$

$$\mathcal{E}^s(S^0(\tilde{z}_{\varepsilon,a_i})) \leq \mu(a_i)\mathcal{E}^s(S^0(z)) + a_i,$$

and

$$\|\nabla \tilde{z}_{\varepsilon,a_i} - \nabla z\|_{L^p(\Omega;M^{2\times 2})} \leq a_i, \qquad \|\tilde{z}_{\varepsilon,a_i} - z\|_{L^q(\Omega;\mathbb{R}^2)} \leq a_i.$$

Setting

$$z_{\varepsilon,a} := \begin{cases} \tilde{z}_{\varepsilon,a_i} & \varepsilon_{i+1} < \varepsilon \leq \varepsilon_i,\ a \leq a_i, \\ \tilde{z}_{\varepsilon,a_{j-1}} & \varepsilon_{i+1} < \varepsilon \leq \varepsilon_i,\ a_j < a \leq a_{j-1},\, j \leq i, \end{cases}$$

we have that

$$\lim_{\varepsilon,a\to 0} \nabla z_{\varepsilon,a} = \nabla z \qquad \text{strongly in } L^p(\Omega; M^{2\times 2}),$$
$$\lim_{\varepsilon,a\to 0} z_{\varepsilon,a} = z \qquad \text{strongly in } L^q(\Omega;\mathbb{R}^2),$$

and

$$\limsup_{\varepsilon,a\to 0} \mathcal{E}^s(S^0(z_{\varepsilon,a})) \leq \mathcal{E}^s(S^0(z)).$$

Since by Theorem 3.2 we have $\mathcal{E}^s(S^0(z_{\varepsilon,a})) \le \liminf_{\varepsilon,a\to 0} \mathcal{E}^s(S^0(z_{\varepsilon,a}))$, we conclude that

$$\lim_{\varepsilon,a\to 0} \mathcal{E}^s(S^0(z_{\varepsilon,a})) = \mathcal{E}^s(S^0(z)).$$

Let us set for every ε, a

$$\Gamma^0_{\varepsilon,a} := S^0(z_{\varepsilon,a}).$$

We have that

$$\lim_{\varepsilon,a\to 0} \mathcal{E}^s(\Gamma^0_{\varepsilon,a}) = \mathcal{E}^s(\Gamma^0).$$

Let us come to the second part of the proof. Let us consider $(\varepsilon_n, a_n)_{n\in\mathbb{N}}$ such that $\varepsilon_n \to 0$ and $a_n \to 0$. If we prove that $(F_{\varepsilon_n,a_n})_{n\in\mathbb{N}}$ Γ-converges to F in the strong topology of $L^1(\Omega;\mathbb{R}^2)$, the proposition is proved since the sequence is arbitrary. Since we can reason up to subsequences, it is not restrictive to assume $a_n \searrow 0$.

Let us start with the Γ-limsup inequality considering $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$, with $S(v) \subseteq \overline{\Omega}_B$. For any n fixed, by Proposition 4.2 there exists $\tilde{v}_{\varepsilon,a_n} \in \mathcal{A}^B_{\varepsilon,a_n}(\Omega;\mathbb{R}^2)$ such that for $\varepsilon \to 0$

$$\nabla \tilde{v}_{\varepsilon,a_n} \to \nabla v \qquad \text{strongly in } L^p(\Omega; M^{2\times 2}),$$

$$\tilde{v}_{\varepsilon,a_n} \to v \qquad \text{strongly in } L^q(\Omega;\mathbb{R}^2),$$

and such that

$$\limsup_{\varepsilon\to 0} \mathcal{E}^s(S^{g_\varepsilon}(\tilde{v}_{\varepsilon,a_n}) \setminus \Gamma^0_{\varepsilon,a_n}) \le \mu(a_n)\mathcal{E}^s(S^{g_\varepsilon}(v) \setminus \Gamma^0)$$

with $\mu(a) \to 1$ as $a \to 0$. For every $m \in \mathbb{N}$ let ε^m be such that for all $\varepsilon \le \varepsilon^m$

$$\mathcal{E}^s(S^{g_\varepsilon}(\tilde{v}_{\varepsilon,a_m}) \setminus \Gamma^0_{\varepsilon,a_m}) \le \mu(a_m)\mathcal{E}^s(S^g(v) \setminus \Gamma^0) + a_m,$$

and

$$\|\nabla \tilde{v}_{\varepsilon,a_m} - \nabla v\|_{L^p(\Omega;M^{2\times 2})} \le a_m, \qquad \|\tilde{v}_{\varepsilon,a_m} - v\|_{L^q(\Omega;\mathbb{R}^2)} \le a_m.$$

We can assume $\varepsilon^m \searrow 0$. Setting

$$v_{\varepsilon_n,a_n} := \begin{cases} \tilde{v}_{\varepsilon_n,a_m} & \varepsilon^{m+1} < \varepsilon_n \le \varepsilon^m, \ n \ge m, \\ \tilde{v}_{\varepsilon_n,a_n} & \varepsilon^{m+1} < \varepsilon_n \le \varepsilon^m, \ n < m, \end{cases}$$

we have that

$$\lim_n \nabla v_{\varepsilon_n,a_n} = \nabla v \quad \text{strongly in } L^p(\Omega; M^{2\times 2}),$$

$$\lim_n v_{\varepsilon_n,a_n} = v \quad \text{strongly in } L^q(\Omega;\mathbb{R}^2),$$

and

$$\limsup_n \mathcal{E}^s(S^{g_\varepsilon}(v_{\varepsilon_n,a_n}) \setminus \Gamma^0_{\varepsilon,a}) \le \mathcal{E}^s(S^g(v) \setminus \Gamma^0).$$

Then we get

$$\begin{aligned} \limsup_n F_{\varepsilon_n,a_n}(v_{\varepsilon_n,a_n}) &\le \limsup_n \mathcal{E}^{el}(0)(v_{\varepsilon_n,a_n}) + \limsup_n \mathcal{E}^s(S^{g_{\varepsilon_n}}(v_{\varepsilon_n,a_n}) \setminus \Gamma^0_{\varepsilon_n,a_n}) \\ &\le \mathcal{E}^{el}(0)(v) + \mathcal{E}^s(S^g(v) \setminus \Gamma^0) = F(v), \end{aligned}$$

so that the Γ-limsup inequality holds.

Let us come to the Γ-liminf inequality. Let $v_n, v \in L^1(\Omega;\mathbb{R}^2)$ be such that $v_n \to v$ strongly in $L^1(\Omega;\mathbb{R}^2)$ and $\liminf_n F_{\varepsilon_n,a_n}(v_n) < +\infty$. By Theorem 3.2, we have $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ with $S(v) \,\tilde{\subset}\, \overline{\Omega}_B$ and

$$\mathcal{E}^{el}(0)(v) \le \liminf_n \mathcal{E}^{el}(0)(v_n).$$

Let us consider Ω' defined in (4.1). Let us extend g_{ε_n} and g to $W^{1,p}(\Omega';\mathbb{R}^2) \cap L^q(\Omega';\mathbb{R}^2)$ in such a way that $g_{\varepsilon_n} \to g$ strongly in $W^{1,p}(\Omega';\mathbb{R}^2) \cap L^q(\Omega';\mathbb{R}^2)$, and let us also extend v_n, v to Ω' setting $v_n = g_{\varepsilon_n}$ and $v = g$ on Ω_D. We indicate these extensions with w_n and w respectively. Let us also set $z_{\varepsilon_n,a_n} = z = 0$ on Ω_D, where z_{ε_n,a_n} and z are such that $\Gamma^0_{\varepsilon,a} = z_{\varepsilon_n,a_n}$ and $\Gamma^0 = S(z)$. We indicate these extension by $\zeta_{\varepsilon_n,a_n}$ and ζ respectively. Then for every $\eta > 0$ we have by Theorem 3.2

$$\mathcal{E}^s(S(w + \eta\zeta)) \le \liminf_n \mathcal{E}^s(S(w_n + \eta\zeta_{\varepsilon_n,a_n})).$$

Since for a.e. $\eta > 0$ we have $S(w+\eta\zeta) = S(w) \cup S(\zeta)$ and $S(w_n + \eta\zeta_{\varepsilon_n,a_n}) = S(w_n) \cup S(\zeta_{\varepsilon_n,a_n})$, we deduce that

$$\mathcal{E}^s(S^g(v) \cup \Gamma^0) \le \liminf_n \mathcal{E}^s(S^{g_{\varepsilon_n}}(v_n) \cup \Gamma^0_{\varepsilon_n,a_n}).$$

Since by assumption $\mathcal{E}^s(\Gamma^0_{\varepsilon_n,a_n}) \to \mathcal{E}^s(\Gamma^0)$, we conclude that

$$\mathcal{E}^s(S^g(v) \setminus \Gamma^0) \le \liminf_n \mathcal{E}^s(S^{g_{\varepsilon_n}}(v_n) \setminus \Gamma^0_{\varepsilon_n,a_n}).$$

We deduce that

$$\mathcal{E}^{el}(0)(v) + \mathcal{E}^s(S^g(v) \setminus \Gamma^0) \le \liminf_n \left[\mathcal{E}^{el}(0)(v_n) + \mathcal{E}^s(S^{g_{\varepsilon_n}}(v_n) \setminus \Gamma^0_{\varepsilon_n,a_n})\right]$$

that is

$$F(v) \le \liminf_n F_{\varepsilon_n,a_n}(v_n).$$

The Γ-liminf inequality holds, and so the proof is concluded. $\square$

6. The discontinuous finite element approximation

In this section we construct a discrete approximation of the quasistatic evolution of brittle fractures proposed in [13] and described in the Preliminaries: the discretization is done both in space and time. Let us consider

$$g_\varepsilon \in W^{1,1}([0,T]; W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)), \quad g_\varepsilon(t) \in \mathcal{AF}_\varepsilon(\Omega;\mathbb{R}^2) \text{ for all } t \in [0,T],$$

where $\mathcal{AF}_\varepsilon(\Omega;\mathbb{R}^2)$ is defined in (4.2). Let $\delta > 0$, and let N_δ be the largest integer such that $\delta(N_\delta - 1) < T$; we set $t^\delta_i := i\delta$ for $0 \le i \le N_\delta - 1$, $t^\delta_{N_\delta} := T$ and $g^{\delta,i}_\varepsilon := g_\varepsilon(t^\delta_i)$. Let $\Gamma^0 \in \mathbf{\Gamma}_{\varepsilon,a}(\Omega)$ be a preexisting crack in Ω, where $\mathbf{\Gamma}_{\varepsilon,a}(\Omega)$ is defined in (5.2).

Proposition 6.1. *Let $\varepsilon > 0$, $a \in]0, \frac{1}{2}[$ and $\delta > 0$ be fixed. Then for all $i = 0, \dots, N_\delta$ there exists $u^{\delta,i}_{\varepsilon,a} \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$ such that, setting*

$$\Gamma^{\delta,i}_{\varepsilon,a} := \Gamma^0 \cup \bigcup_{r=0}^{i} S^{g^{\delta,r}_\varepsilon}(u^{\delta,r}_{\varepsilon,a}),$$

we have for all $v \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$

$$\mathcal{E}^{el}(0)(u^{\delta,0}_{\varepsilon,a}) + \mathcal{E}^s\left(S^{g^{\delta,0}_\varepsilon}(u^{\delta,0}_{\varepsilon,a}) \setminus \Gamma^0\right) \le \mathcal{E}^{el}(0)(v) + \mathcal{E}^s\left(S^{g^{\delta,0}_\varepsilon}(v) \setminus \Gamma^0\right), \tag{6.1}$$

and for $1 \le i \le N_\delta$

$$\mathcal{E}^{el}(t^\delta_i)(u^{\delta,i}_{\varepsilon,a}) + \mathcal{E}^s\left(S^{g^{\delta,i}_\varepsilon}(u^{\delta,i}_{\varepsilon,a}) \setminus \Gamma^{\delta,i-1}_{\varepsilon,a}\right) \le \mathcal{E}^{el}(t^\delta_i)(v) + \mathcal{E}^s\left(S^{g^{\delta,i}_\varepsilon}(v) \setminus \Gamma^{\delta,i-1}_{\varepsilon,a}\right). \tag{6.2}$$

Proof. Let $u^{\delta,0}_{\varepsilon,a}$ be a minimum of the following problem

$$\min_{u \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)} \left\{\mathcal{E}^{el}(0)(u) + \mathcal{E}^s(S^{g^{\delta,0}_\varepsilon}(u) \setminus \Gamma^0)\right\}. \tag{6.3}$$

We set $\Gamma^{\delta,0}_{\varepsilon,a} := \Gamma^0 \cup S^{g^{\delta,0}_\varepsilon}(u^{\delta,0}_{\varepsilon,a})$. Recursively, supposing to have constructed $u^{\delta,i-1}_{\varepsilon,a}$ and $\Gamma^{\delta,i-1}_{\varepsilon,a}$, let $u^{\delta,i}_{\varepsilon,a}$ be a minimum for

$$\min_{u \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)} \left\{\mathcal{E}^{el}(t^\delta_i)(u) + \mathcal{E}^s(S^{g^{\delta,i}_\varepsilon}(u) \setminus \Gamma^{\delta,i-1}_{\varepsilon,a})\right\}. \tag{6.4}$$

We set $\Gamma^{\delta,i}_{\varepsilon,a} := \Gamma^{\delta,i-1}_{\varepsilon,a} \cup S^{g^{\delta,i}_\varepsilon}(u^{\delta,i}_{\varepsilon,a})$. It is clear by construction that (6.1) and (6.2) hold.

Let us prove that problem (6.4) admits a solution, problem (6.3) being similar. Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence for problem (6.4): since $g^{\delta,i}_\varepsilon$ is an admissible test function, we deduce that for n large

$$\mathcal{E}^{el}(t^\delta_i)(u_n) + \mathcal{E}^s\left(S^{g^{\delta,i}_\varepsilon}(u_n) \setminus \Gamma^{\delta,i-1}_{\varepsilon,a}\right) \le \mathcal{E}^{el}(t^\delta_i)(g^{\delta,i}_\varepsilon) + 1.$$

By the lower estimate on the elastic energy (3.20), we deduce that for n large

$$\alpha^{\mathcal{E}}_0 \left(\|\nabla u_n\|^p_p + \|u_n\|^q_q\right) + \mathcal{E}^s\left(S^{g^{\delta,i}_\varepsilon}(u_n) \setminus \Gamma^{\delta,i-1}_{\varepsilon,a}\right) \le \mathcal{E}^{el}(t^\delta_i)(g^{\delta,i}_\varepsilon) + 1 + \beta^{\mathcal{E}}_0. \tag{6.5}$$

Let us indicate by $T^1_n, \dots, T^k_n$ the triangles of $\mathbf{T}(u_n)$. Up to a subsequence, there exists $\mathbf{T} = \{T^1, \dots, T^k\} \in \mathcal{T}_{\varepsilon,a}(\Omega)$ such that for all $i = 1, \dots, k$ we have $T^i_n \to T^i$ in the Hausdorff metric (see Section 2 for a precise definition). Let us consider $T^i \in \mathbf{T}$, and let $\tilde{T}^i$ be contained in the interior of T^i. For n large enough, $\tilde{T}^i$ is contained in the interior of T^i_n; moreover $(u_n)_{|\tilde{T}^i}$ is affine and by (6.5) we have $\int_{\tilde{T}^i} |\nabla u_n|^p \, dx + \|u_n\|_{L^\infty(\tilde{T}^i;\mathbb{R}^2)} \le C$, with C independent of n. We deduce that there exists a function u^i affine on $\tilde{T}^i$ such that up to a subsequence $u_n \to u$ uniformly on $\tilde{T}^i$. Since $\tilde{T}^i$ is arbitrary, it turns out that u^i is actually defined on T^i and

$$\mathcal{E}^{el}(t^\delta_i)_{|T^i}(u^i) \le \liminf_n \mathcal{E}^{el}(t^\delta_i)_{|T^i}(u^i_n),$$

where $\mathcal{E}^{el}(t^\delta_i)_{|T^i}$ denotes the restriction of $\mathcal{E}^{el}(t^\delta_i)$ to the maps defined on T^i. Let $u \in \mathcal{A}_{\varepsilon,a}(\Omega;\mathbb{R}^2)$ be such that $u = u^i$ on T^i for every $i = 1, \dots, k$: we have

$$\mathcal{E}^{el}(t^\delta_i)(u) \le \liminf_n \mathcal{E}^{el}(t^\delta_i)(u_n).$$

On the other hand, it is easy to see that $S^{g^{\delta,i}_\varepsilon}(u)$ is contained in the Hausdorff limit of $S^{g^{\delta,i}_\varepsilon}(u_n)$, and so $u \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$; moreover we deduce

$$\mathcal{E}^s\left(S^{g^{\delta,i}_\varepsilon}(u) \setminus \Gamma^{\delta,i-1}_{\varepsilon,a}\right) \le \liminf_n \mathcal{E}^s\left(S^{g^{\delta,i}_\varepsilon}(u_n) \setminus \Gamma^{\delta,i-1}_{\varepsilon,a}\right).$$

We conclude that u is a minimum point for the problem (6.4), so that the proof is concluded. □

The following estimate on the total energy is essential in order to study the asymptotic behavior of the discrete evolution as $\delta \to 0$, $\varepsilon \to 0$ and $a \to 0$. Let us set $u^\delta_{\varepsilon,a}(t) := u^{\delta,i}_{\varepsilon,a}$ for all $t^\delta_i \le t < t^\delta_{i+1}$ and $i = 0, \dots, N_\delta - 1$, $u^\delta_{\varepsilon,a}(T) = u^{\delta,N_\delta}_{\varepsilon,a}$.

Proposition 6.2. *For all $0 \le j \le i \le N_\delta$ we have*

$$\begin{aligned}(6.6)\quad \mathcal{E}(t^\delta_i)(u^{\delta,i}_{\varepsilon,a}, \Gamma^{\delta,i}_{\varepsilon,a}) &\le \mathcal{E}(t^\delta_j)(u^{\delta,j}_{\varepsilon,a}, \Gamma^{\delta,j}_{\varepsilon,a}) + \int_{t^\delta_j}^{t^\delta_i} \langle \partial\mathcal{W}(\nabla u^\delta_{\varepsilon,a}(\tau)), \nabla \dot{g}_\varepsilon(\tau)\rangle \, d\tau \\ &- \int_{t^\delta_j}^{t^\delta_i} \dot{\mathcal{F}}(\tau)(u^\delta_{\varepsilon,a}(\tau)) \, d\tau - \int_{t^\delta_j}^{t^\delta_i} \langle \partial\mathcal{F}(\tau)(u^\delta_{\varepsilon,a}(\tau)), \dot{g}_\varepsilon(\tau)\rangle \, d\tau \\ &- \int_{t^\delta_j}^{t^\delta_i} \dot{\mathcal{G}}(\tau)(u^\delta_{\varepsilon,a}(\tau)) \, d\tau - \int_{t^\delta_j}^{t^\delta_i} \langle \partial\mathcal{G}(\tau)(u^\delta_{\varepsilon,a}(\tau)), \dot{g}_\varepsilon(\tau)\rangle \, d\tau + e^\delta_{\varepsilon,a},\end{aligned}$$

where $e^\delta_{\varepsilon,a} \to 0$ as $\delta \to 0$ uniformly in ε and a.

Proof. By the minimality property (6.2), comparing $u^{\delta,i}_{\varepsilon,a}$ with $u^{\delta,i-1}_{\varepsilon,a} - g^{\delta,i-1}_\varepsilon + g^{\delta,i}_\varepsilon$ we get

$$\begin{aligned}(6.7)\quad &\mathcal{W}(\nabla u^{\delta,i}_{\varepsilon,a}) - \mathcal{F}(t^\delta_i)(u^{\delta,i}_{\varepsilon,a}) - \mathcal{G}(t^\delta_i)(u^{\delta,i}_{\varepsilon,a}) + \mathcal{E}^s(S^{g^\delta_i}(u^{\delta,i}_{\varepsilon,a}) \setminus \Gamma^\delta_{i-1}) \\ &\le \mathcal{W}(\nabla u^{\delta,i-1}_{\varepsilon,a} - \nabla g^{\delta,i-1}_\varepsilon + \nabla g^{\delta,i}_\varepsilon) - \mathcal{F}(t^\delta_i)(u^{\delta,i-1}_{\varepsilon,a} - g^{\delta,i-1}_\varepsilon + g^{\delta,i}_\varepsilon) \\ &\qquad - \mathcal{G}(t^\delta_i)(u^{\delta,i-1}_{\varepsilon,a} - g^{\delta,i-1}_\varepsilon + g^{\delta,i}_\varepsilon).\end{aligned}$$

We have

$$\begin{aligned}(6.8)\quad \mathcal{W}(\nabla u^{\delta,i-1}_{\varepsilon,a} - \nabla g^{\delta,i-1}_\varepsilon + \nabla g^{\delta,i}_\varepsilon) &= \mathcal{W}(\nabla u^{\delta,i-1}_{\varepsilon,a}) \\ &+ \langle \partial\mathcal{W}(\nabla u^{\delta,i-1}_{\varepsilon,a} + \vartheta^{\delta,i-1}_{\varepsilon,a}(\nabla g^{\delta,i}_\varepsilon - \nabla g^{\delta,i-1}_\varepsilon)), \nabla g^{\delta,i}_\varepsilon - \nabla g^{\delta,i-1}_\varepsilon\rangle \\ &= \mathcal{W}(\nabla u^{\delta,i-1}_{\varepsilon,a}) + \int_{t^\delta_{i-1}}^{t^\delta_i} \langle \partial\mathcal{W}(\nabla u^\delta_{\varepsilon,a}(\tau) + v^\delta_{\varepsilon,a}(\tau)), \nabla \dot{g}_\varepsilon(\tau)\rangle \, d\tau,\end{aligned}$$

where $\vartheta^{\delta,i-1}_{\varepsilon,a} \in]0,1[$ and $v^\delta_{\varepsilon,a}(\tau) := \vartheta^{\delta,i-1}_{\varepsilon,a}(\nabla g^{\delta,i}_\varepsilon - \nabla g^{\delta,i-1}_\varepsilon)$ for all $\tau \in [t^\delta_{i-1}, t^\delta_i[$.

Similarly we obtain

$$(6.9)\quad \mathcal{F}(t^\delta_i)(u^{\delta,i-1}_{\varepsilon,a} - g^{\delta,i-1}_\varepsilon + g^{\delta,i}_\varepsilon) = \mathcal{F}(t^\delta_i)(u^{\delta,i-1}_{\varepsilon,a}) + \int_{t^\delta_{i-1}}^{t^\delta_i} \langle \partial\mathcal{F}(t^\delta_i)(u^\delta_{\varepsilon,a}(\tau) + w^\delta_{\varepsilon,a}(\tau)), \dot{g}_\varepsilon(\tau)\rangle \, d\tau,$$

and

$$\mathcal{G}(t_i^\delta)(u_{\varepsilon,a}^{\delta,i-1} - g_\varepsilon^{\delta,i-1} + g_\varepsilon^{\delta,i}) = \mathcal{G}(t_i^\delta)(u_{\varepsilon,a}^{\delta,i-1}) + \int_{t_{i-1}^\delta}^{t_i^\delta} \langle \partial\mathcal{G}(t_i^\delta)(u_{\varepsilon,a}^\delta(\tau) + z_{\varepsilon,a}^\delta(\tau)), \dot{g}_\varepsilon(\tau)\rangle \, d\tau, \tag{6.10}$$

where $w_{\varepsilon,a}^\delta(\tau) := \lambda_{\varepsilon,a}^{\delta,i-1}(g_\varepsilon^{\delta,i} - g_\varepsilon^{\delta,i-1})$, $z_{\varepsilon,a}^\delta(\tau) := \nu_{\varepsilon,a}^{\delta,i-1}(g_\varepsilon^{\delta,i} - g_\varepsilon^{\delta,i-1})$ for all $\tau \in [t_{i-1}^\delta, t_i^\delta[$, and $\lambda_{\varepsilon,a}^{\delta,i-1}, \nu_{\varepsilon,a}^{\delta,i-1} \in]0,1[$.

Since by (3.10) we have for $\tau \in [t_{i-1}^\delta, t_i^\delta[$

$$\begin{aligned}\langle \partial\mathcal{F}(t_i^\delta)(u_{\varepsilon,a}^\delta(\tau) + w_{\varepsilon,a}^\delta(\tau)), \dot{g}_\varepsilon(\tau)\rangle - \langle \partial\mathcal{F}(\tau)(u_{\varepsilon,a}^\delta(\tau) + w_{\varepsilon,a}^\delta(\tau)), \dot{g}_\varepsilon(\tau)\rangle \\ = \int_\tau^{t_i^\delta} \langle \partial\dot{\mathcal{F}}(s)(u_{\varepsilon,a}^\delta(\tau) + w_{\varepsilon,a}^\delta(\tau)), \dot{g}_\varepsilon(\tau)\rangle \, ds\end{aligned}$$

we get by (3.13)

$$\begin{aligned}&\big|\langle \partial\mathcal{F}(t_i^\delta)(u_{\varepsilon,a}^\delta(\tau) + w_{\varepsilon,a}^\delta(\tau)), \dot{g}_\varepsilon(\tau)\rangle - \langle \partial\mathcal{F}(\tau)(u_{\varepsilon,a}^\delta(\tau) + w_{\varepsilon,a}^\delta(\tau)), \dot{g}_\varepsilon(\tau)\rangle\big| \\ &\qquad \le \int_\tau^{t_i^\delta} |\langle \partial\dot{\mathcal{F}}(s)(u_{\varepsilon,a}^\delta(\tau) + w_{\varepsilon,a}^\delta(\tau)), \dot{g}_\varepsilon(\tau)\rangle| \, ds \\ &\qquad \le \int_\tau^{t_i^\delta} \left[\alpha_4^{\mathcal{F}}(s)\|u_{\varepsilon,a}^\delta(\tau) + w_{\varepsilon,a}^\delta(\tau)\|_{\dot{q}}^{\dot{q}-1} + \beta_4^{\mathcal{F}}(s)\right] \|\dot{g}_\varepsilon(\tau)\|_{\dot{q}} \, ds \le \gamma_{\mathcal{F}}^{\delta,\varepsilon,a}\|\dot{g}_\varepsilon(\tau)\|_{\dot{q}},\end{aligned} \tag{6.11}$$

where

$$\gamma_{\mathcal{F}}^{\delta,\varepsilon,a} := \max_{1\le i\le N_\delta} \left(\|u_{\varepsilon,a}^{\delta,i-1} + \lambda_{\varepsilon,a}^{\delta,i-1}(g_\varepsilon^{\delta,i} - g_\varepsilon^{\delta,i-1})\|_{\dot{q}-1}^{\dot{q}} \int_{t_{i-1}^\delta}^{t_i^\delta} \alpha_4^{\mathcal{F}}(s) \, ds + \int_{t_{i-1}^\delta}^{t_i^\delta} \beta_4^{\mathcal{F}}(s) \, ds \right).$$

Similarly we obtain

$$\big|\langle \partial\mathcal{G}(t_i^\delta)(u_{\varepsilon,a}^\delta(\tau) + z_{\varepsilon,a}^\delta(\tau)), \dot{g}_\varepsilon(\tau)\rangle - \langle \partial\mathcal{G}(\tau)(u_{\varepsilon,a}^\delta(\tau) + z_{\varepsilon,a}^\delta(\tau)), \dot{g}_\varepsilon(\tau)\rangle\big| \le \gamma_{\mathcal{G}}^{\delta,\varepsilon,a}\|\dot{g}_\varepsilon(\tau)\|_{r,\partial_S\Omega}, \tag{6.12}$$

where

$$\gamma_{\mathcal{G}}^{\delta,\varepsilon,a} := \max_{1\le i\le N_\delta} \left(\|u_{\varepsilon,a}^{\delta,i-1} + \nu_{\varepsilon,a}^{\delta,i-1}(g_\varepsilon^{\delta,i} - g_\varepsilon^{\delta,i-1})\|_{r,\partial_S\Omega}^{r-1} \int_{t_{i-1}^\delta}^{t_i^\delta} a_4^{\mathcal{G}}(s) \, ds + \int_{t_{i-1}^\delta}^{t_i^\delta} b_4^{\mathcal{G}}(s) \, ds \right).$$

From (6.7), taking into account (6.8), (6.9), (6.10), (6.11), (6.12), we have

$$\begin{aligned}\mathcal{E}(t_i^\delta)(u_{\varepsilon,a}^{\delta,i}, \Gamma_{\varepsilon,a}^{\delta,i}) \le{}& \mathcal{E}(t_{i-1}^\delta)(u_{\varepsilon,a}^{\delta,i-1}, \Gamma_{\varepsilon,a}^{\delta,i-1}) + \int_{t_{i-1}^\delta}^{t_i^\delta} \langle \partial\mathcal{W}(\nabla u_{\varepsilon,a}^\delta(\tau) + v_{\varepsilon,a}^\delta(\tau)), \nabla\dot{g}_\varepsilon(\tau)\rangle \, d\tau \\ &- \int_{t_{i-1}^\delta}^{t_i^\delta} \dot{\mathcal{F}}(\tau)(u_{\varepsilon,a}^\delta(\tau)) \, d\tau - \int_{t_{i-1}^\delta}^{t_i^\delta} \langle \partial\mathcal{F}(\tau)(u_{\varepsilon,a}^\delta(\tau) + w_{\varepsilon,a}^\delta(\tau)), \dot{g}_\varepsilon(\tau)\rangle \, d\tau \\ &- \int_{t_{i-1}^\delta}^{t_i^\delta} \dot{\mathcal{G}}(\tau)(u_{\varepsilon,a}^\delta(\tau)) \, d\tau - \int_{t_{i-1}^\delta}^{t_i^\delta} \langle \partial\mathcal{G}(\tau)(u_{\varepsilon,a}^\delta(\tau) + z_{\varepsilon,a}^\delta(\tau)), \dot{g}_\varepsilon(\tau)\rangle \, d\tau \\ &+ \gamma_{\mathcal{F}}^{\delta,\varepsilon,a} \int_{t_{i-1}^\delta}^{t_i^\delta} \|\dot{g}_\varepsilon(\tau)\|_{\dot{q}} \, d\tau + \gamma_{\mathcal{G}}^{\delta,\varepsilon,a} \int_{t_{i-1}^\delta}^{t_i^\delta} \|\dot{g}_\varepsilon(\tau)\|_{r,\partial_S\Omega} \, d\tau.\end{aligned} \tag{6.13}$$

Taking now $0 \le j \le i \le N_\delta$, summing in (6.13) form t_j^δ to t_i^δ, we obtain

$$\mathcal{E}(t_i^\delta)(u_{\varepsilon,a}^{\delta,i},\Gamma_{\varepsilon,a}^{\delta,i}) \le \mathcal{E}(t_j^\delta)(u_{\varepsilon,a}^{\delta,j},\Gamma_{\varepsilon,a}^{\delta,j}) + \int_{t_j^\delta}^{t_i^\delta} \langle \partial\mathcal{W}(\nabla u_{\varepsilon,a}^\delta(\tau) + v_{\varepsilon,a}^\delta(\tau)), \nabla \dot g_\varepsilon(\tau)\rangle \, d\tau \tag{6.14}$$
$$- \int_{t_j^\delta}^{t_i^\delta} \dot{\mathcal{F}}(\tau)(u_{\varepsilon,a}^\delta(\tau))\, d\tau - \int_{t_j^\delta}^{t_i^\delta} \langle \partial\mathcal{F}(\tau)(u_{\varepsilon,a}^\delta(\tau) + w_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle \, d\tau$$
$$- \int_{t_j^\delta}^{t_i^\delta} \dot{\mathcal{G}}(\tau)(u_{\varepsilon,a}^\delta(\tau))\, d\tau - \int_{t_j^\delta}^{t_i^\delta} \langle \partial\mathcal{G}(\tau)(u_{\varepsilon,a}^\delta(\tau) + z_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle \, d\tau$$
$$+ \gamma_{\mathcal{F}}^{\delta,\varepsilon,a} \int_{t_j^\delta}^{t_i^\delta} \|\dot g_\varepsilon(\tau)\|_{\dot q}\, d\tau + \gamma_{\mathcal{G}}^{\delta,\varepsilon,a} \int_{t_j^\delta}^{t_i^\delta} \|\dot g_\varepsilon(\tau)\|_{r,\partial_S\Omega}\, d\tau.$$

Setting

$$e_{\varepsilon,a}^\delta := \int_0^1 |\langle \partial\mathcal{W}(\nabla u_{\varepsilon,a}^\delta(\tau) + v_{\varepsilon,a}^\delta(\tau)), \nabla \dot g_\varepsilon(\tau)\rangle - \langle \partial\mathcal{W}(\nabla u_{\varepsilon,a}^\delta(\tau)), \nabla \dot g_\varepsilon(\tau)\rangle| \, d\tau \tag{6.15}$$
$$+ \int_0^1 |\langle \partial\mathcal{F}(\tau)(u_{\varepsilon,a}^\delta(\tau) + w_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle - \langle \partial\mathcal{F}(\tau)(u_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle| \, d\tau$$
$$+ \int_0^1 |\langle \partial\mathcal{G}(\tau)(u_{\varepsilon,a}^\delta(\tau) + z_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle - \langle \partial\mathcal{G}(\tau)(u_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle| \, d\tau$$
$$+ \gamma_{\mathcal{F}}^{\delta,\varepsilon,a} \int_0^1 \|\dot g_\varepsilon(\tau)\|_{\dot q}\, d\tau + \gamma_{\mathcal{G}}^{\delta,\varepsilon,a} \int_0^1 \|\dot g_\varepsilon(\tau)\|_{r,\partial_S\Omega}\, d\tau,$$

from (6.14) we formally obtain (6.6). Let us prove that $e_{|eps,a}^\delta \to 0$ as $\delta \to 0$ uniformly in ε and a. By (6.2), comparing $u_{\varepsilon,a}^{\delta,i}$ with $g_\varepsilon^{\delta,i}$, and taking into account (3.20), we get for all $i = 1, \dots, N_\delta$,

$$\|\nabla u_{\varepsilon,a}^{\delta,i}\|_p + \|u_{\varepsilon,a}^{\delta,i}\|_q \le C',$$

where

$$C' := \frac{1}{\alpha_0^{\mathcal{E}}} \max_{i=0,\dots,N_\delta} \left(\mathcal{E}^{el}(t_i^\delta)(g_\varepsilon^{\delta,i}) + \beta_0^{\mathcal{E}} \right).$$

Since Ω_S is Lipschitz, there exists $K_S > 0$ depending only on p, q such that

$$\|u\|_{p,\Omega_S} \le K_S(\|\nabla u\|_{p,\Omega_S} + \|u\|_{q,\Omega_S})$$

for all $u \in W^{1,p}(\Omega_S;\mathbb{R}^2) \cap L^q(\Omega_S;\mathbb{R}^2)$. Taking into account (3.15), we obtain

$$\|u_{\varepsilon,a}^{\delta,i}\|_{r,\partial_S\Omega} \le C''$$

for some C'' independent of δ. Since $g_\varepsilon \in W^{1,1}([0,T]; W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2))$, we obtain that for all $\tau \in [0,T]$ as $\delta \to 0$

$$v_{\varepsilon,a}^\delta(\tau) \to 0 \text{ strongly in } L^p(\Omega; M^{2\times 2}),$$
$$w_{\varepsilon,a}^\delta(\tau) \to 0 \text{ strongly in } L^q(\Omega; \mathbb{R}^2),$$
$$z_{\varepsilon,a}^\delta(\tau) \to 0 \text{ strongly in } L^r(\partial_S\Omega; \mathbb{R}^2).$$

Moreover $\gamma_{\mathcal{F}}^{\delta,\varepsilon,a} \to 0$ and $\gamma_{\mathcal{G}}^{\delta,\varepsilon,a} \to 0$ as $\delta \to 0$. Finally, by [13, Lemma 4.9], as $\delta \to 0$ we have that for all $\tau \in [0,T]$

$$|\langle \partial\mathcal{W}(\nabla u_{\varepsilon,a}^\delta(\tau) + v_{\varepsilon,a}^\delta(\tau)), \nabla \dot g_\varepsilon(\tau)\rangle - \langle \partial\mathcal{W}(\nabla u_{\varepsilon,a}^\delta(\tau)), \nabla \dot g_\varepsilon(\tau)\rangle| \to 0,$$
$$|\langle \partial\mathcal{F}(\tau)(u_{\varepsilon,a}^\delta(\tau) + w_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle - \langle \partial\mathcal{F}(\tau)(u_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle| \to 0,$$
$$|\langle \partial\mathcal{G}(\tau)(u_{\varepsilon,a}^\delta(\tau) + z_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle - \langle \partial\mathcal{G}(\tau)(u_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle| \to 0,$$

uniformly in ε, a. By the Dominated Convergence Theorem, we conclude that $e_{\varepsilon,a}^\delta \to 0$ as $\delta \to 0$ uniformly in ε and a, and the proof is finished. $\square$

7. The approximation result

In this section we study the asymptotic behavior of the discrete evolution obtained in Section 6. Let us consider a given initial crack $\Gamma^0 \in \mathbf{\Gamma}(\Omega)$ where $\mathbf{\Gamma}(\Omega)$ is defined as in (5.1), and a boundary deformation $g \in W^{1,1}\big([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)\big)$. Let $\Gamma^0_{\varepsilon,a} \in \mathbf{\Gamma}_{\varepsilon,a}(\Omega)$ be an approximation of Γ^0 in the sense of Proposition 5.1, and let us consider

$$g_\varepsilon \in W^{1,1}\big([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)\big),$$

such that

$$g_\varepsilon(t) \in \mathcal{AF}_\varepsilon(\Omega;\mathbb{R}^2) \text{ for all } t \in [0,T],$$

and such that for $\varepsilon \to 0$

$$g_\varepsilon \to g \quad \text{strongly in } W^{1,1}([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)).$$

Let

$$\{(u^{\delta,i}_{\varepsilon,a}, \Gamma^{\delta,i}_{\varepsilon,a}),\, i = 0,\dots,N_\delta\}$$

be the discrete evolution relative to the initial crack $\Gamma^0_{\varepsilon,a}$ and boundary data g_ε given by Proposition 6.1. We make the following piecewise constant interpolation in time:

$$u^\delta_{\varepsilon,a}(t) := u^{\delta,i}_{\varepsilon,a}, \qquad \Gamma^\delta_{\varepsilon,a}(t) := \Gamma^{\delta,i}_{\varepsilon,a}, \qquad g^\delta_\varepsilon(t) := g_\varepsilon(t^\delta_i) \qquad \text{for } t^\delta_i \le t < t^\delta_{i+1}, \tag{7.1}$$

$i = 0,\dots,N_\delta - 1$, and $u^\delta_{\varepsilon,a}(T) := u^{\delta,N_\delta}_{\varepsilon,a}$, $\Gamma^\delta_{\varepsilon,a}(T) := \Gamma^{\delta,N_\delta}_{\varepsilon,a}$, $g^\delta_\varepsilon(T) := g_\varepsilon(T)$.

By Proposition 6.2, for all $v \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$ we have

$$\mathcal{E}^{el}(0)(u^\delta_{\varepsilon,a}(0)) + \mathcal{E}^s\left(S^{g^\delta_\varepsilon(0)}(u^\delta_{\varepsilon,a}(0)) \setminus \Gamma^0_{\varepsilon,a}\right) \le \mathcal{E}^{el}(0)(v) + \mathcal{E}^s\left(S^{g^\delta_\varepsilon(0)}(v) \setminus \Gamma^0_{\varepsilon,a}\right),$$

and for all $t \in [t^\delta_i, t^\delta_{i+1}[$ and for all $v \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$

$$\mathcal{E}^{el}(t^\delta_i)(u^\delta_{\varepsilon,a}(t)) \le \mathcal{E}^{el}(t^\delta_i)(v) + \mathcal{E}^s\left(S^{g^\delta_\varepsilon(t)}(v) \setminus \Gamma^\delta_{\varepsilon,a}(t)\right). \tag{7.2}$$

Here $\mathcal{E}^{el}$ and $\mathcal{E}^s$ are defined in (3.19) and (3.1) respectively. Finally for all $0 \le s \le t \le 1$ we have

$$\begin{aligned}\mathcal{E}(t^\delta_i)(u^\delta_{\varepsilon,a}(t), \Gamma^\delta_{\varepsilon,a}(t)) \le{}& \mathcal{E}(s^\delta_j)(u^\delta_{\varepsilon,a}(s), \Gamma^\delta_{\varepsilon,a}(s)) + \int_{s^\delta_j}^{t^\delta_i} \langle \partial\mathcal{W}(\nabla u^\delta_{\varepsilon,a}(\tau)), \nabla \dot g_\varepsilon(\tau)\rangle\, d\tau \\ &- \int_{s^\delta_j}^{t^\delta_i} \dot{\mathcal{F}}(\tau)(u^\delta_{\varepsilon,a}(\tau)) - \int_{s^\delta_j}^{t^\delta_i} \langle \partial\mathcal{F}(\tau)(u^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle\, d\tau \\ &- \int_{s^\delta_j}^{t^\delta_i} \dot{\mathcal{G}}(\tau)(u^\delta_{\varepsilon,a}(\tau)) - \int_{s^\delta_j}^{t^\delta_i} \langle \partial\mathcal{G}(\tau)(u^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle\, d\tau + e^\delta_{\varepsilon,a},\end{aligned} \tag{7.3}$$

where $s^\delta_j \le s < s^\delta_{j+1}$ and $t^\delta_i \le t < t^\delta_{i+1}$, $e^\delta_{\varepsilon,a}$ is defined as in (6.15), and $\mathcal{E}(t)(u,\Gamma)$ is as in (3.18). Recall that $e^\delta_{\varepsilon,a} \to 0$ as $\delta \to 0$ uniformly in ε, a.

Comparing $u^\delta_{\varepsilon,a}(t)$ with $g^\delta_\varepsilon(t)$ by (7.2), and in view of (3.7), (3.11), (3.12), (3.16), (3.17), (6.1) and (3.2), by (7.3) with $s = 0$ we deduce that there exists $C' \in]0,+\infty[$ such that for all t, δ, ε and a

$$\|\nabla u^\delta_{\varepsilon,a}(t)\|_p + \|u^\delta_{\varepsilon,a}(t)\|_q + \mathcal{H}^1(\Gamma^\delta_{\varepsilon,a}(t)) \le C'. \tag{7.4}$$

By the time dependence of $\mathcal{E}^{el}(\cdot,\cdot)$, in view of (7.4), by (7.2) and (7.3) we have that there exists $o^\delta_{\varepsilon,a} \to 0$ as $\delta, \varepsilon \to 0$ uniformly in a such that for all $t \in [0,T]$ and for all $v \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$

$$\mathcal{E}^{el}(t)(u^\delta_{\varepsilon,a}(t)) \le \mathcal{E}^{el}(t)(v) + \mathcal{E}^s\left(S^{g^\delta_\varepsilon(t)}(v) \setminus \Gamma^\delta_{\varepsilon,a}(t)\right) + o^\delta_{\varepsilon,a}, \tag{7.5}$$

and for all $0 \le s \le t \le T$

$$\begin{aligned}\mathcal{E}(t)(u^\delta_{\varepsilon,a}(t),\Gamma^\delta_{\varepsilon,a}(t)) \le & \mathcal{E}(s)(u^\delta_{\varepsilon,a}(s),\Gamma^\delta_{\varepsilon,a}(s)) + \int_s^t \langle \partial\mathcal{W}(\nabla u^\delta_{\varepsilon,a}(\tau)), \nabla \dot g_\varepsilon(\tau)\rangle \, d\tau \\ & - \int_s^t \dot{\mathcal{F}}(\tau)(u^\delta_{\varepsilon,a}(\tau)) - \int_s^t \langle \partial\mathcal{F}(\tau)(u^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle \, d\tau \\ & - \int_s^t \dot{\mathcal{G}}(\tau)(u^\delta_{\varepsilon,a}(\tau)) - \int_s^t \langle \partial\mathcal{G}(\tau)(u^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle \, d\tau + o^\delta_{\varepsilon,a}.\end{aligned} \tag{7.6}$$

Inequality (7.4) gives a natural precompactness of $(u^\delta_{\varepsilon,a}(t))$ in $GSBV^p_q(\Omega;\mathbb{R}^2)$. The main result of the paper is the following.

Theorem 7.1. *Let $\delta > 0$, $\varepsilon > 0$, $a \in]0,\frac{1}{2}[$, and let $\{t \to (u^\delta_{\varepsilon,a}(t),\Gamma^\delta_{\varepsilon,a}(t)) : t \in [0,T]\}$ be the discrete evolution given by (7.1) relative to the initial crack $\Gamma^0_{\varepsilon,a}$ and the boundary data g_ε. Then there exist a quasistatic evolution $\{t \to (u(t),\Gamma(t))\}$ in the sense of Theorem 3.1 and sequences $\delta_n \to 0$, $\varepsilon_n \to 0$, $a_n \to 0$, such that setting $u_n(t) := u^{\delta_n}_{\varepsilon_n,a_n}(t)$ and $\Gamma_n(t) := \Gamma^{\delta_n}_{\varepsilon_n,a_n}(t)$, for all $t \in [0,T]$ the following facts hold.*

(a) *For every $t \in [0,T]$, $(u_n(t))_{n\in\mathbb{N}}$ is weakly precompact in $GSBV^p_q(\Omega;\mathbb{R}^2)$, and every accumulation point $\tilde u(t)$ is such that $S^{g(t)}(\tilde u(t)) \,\widetilde{\subset}\, \Gamma(t)$,*

$$\mathcal{E}^{el}(t)(\tilde u(t)) \le \mathcal{E}^{el}(t)(v) + \mathcal{E}^s\left(S^{g(t)}(v) \setminus \Gamma(t)\right) \tag{7.7}$$

for all $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ with $S(v) \,\widetilde{\subset}\, \overline{\Omega}_B$, and

$$\mathcal{E}^{el}(t)(u_n(t)) \to \mathcal{E}^{el}(t)(\tilde u(t)).$$

Moreover there exists a subsequence of $(\delta_n,\varepsilon_n,a_n)_{n\in\mathbb{N}}$ (depending on t) such that

$$u_n(t) \rightharpoonup u(t) \quad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2).$$

(b) *For every $t \in [0,T]$ we have*

$$\mathcal{E}(t)(u_n(t),\Gamma_n(t)) \to \mathcal{E}(t)(u(t),\Gamma(t)); \tag{7.8}$$

more precisely elastic and surface energies converge separately, that is

$$\mathcal{E}^{el}(t)(u_n(t)) \to \mathcal{E}^{el}(t)(u(t)), \qquad \mathcal{E}^s(\Gamma_n(t)) \to \mathcal{E}^s(\Gamma(t)). \tag{7.9}$$

For the proof of Theorem 7.1 we need two preliminary steps. First of all, we fix a and study the asymptotic for $\delta,\varepsilon \to 0$ (Lemma 7.2), and then we let $a \to 0$ using a diagonal argument (Lemma 7.4).

Lemma 7.2. *Let a be fixed, $t \in [0,T]$, and let $\delta_n \to 0$ and $\varepsilon_n \to 0$. There exists $\Gamma_a(t) \in \mathcal{R}(\overline{\Omega}_B;\partial_N\Omega)$ and a subsequence of $(\delta_n,\varepsilon_n)_{n\in\mathbb{N}}$ (which we denote with the same symbol), such that the following facts hold:*

(a) *if $w_n \in \mathcal{A}^B_{\varepsilon_n,a}(\Omega;\mathbb{R}^2)$ is such that $S^{g^{\delta_n}_{\varepsilon_n}(t)}(w_n) \subseteq \Gamma^{\delta_n}_{\varepsilon_n,a}(t)$ and*

$$w_n \rightharpoonup w \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2),$$

then we have

$$S^{g(t)}(w) \,\widetilde{\subset}\, \Gamma_a(t);$$

(b) *there exists $\mu(a)$ with $\mu(a) \to 1$ as $a \to 0$ such that for every accumulation point $u_a(t)$ of $(u^{\delta_n}_{\varepsilon_n,a}(t))_{n\in\mathbb{N}}$ for the weak convergence in $GSBV^p_q(\Omega;\mathbb{R}^2)$ and for all $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ with $S(v) \,\widetilde{\subset}\, \overline{\Omega}_B$, we have*

$$\mathcal{E}^{el}(t)(u_a(t)) \le \mathcal{E}^{el}(t)(v) + \mu(a)\mathcal{E}^s\left(S^{g(t)}(v) \setminus \Gamma_a(t)\right); \tag{7.10}$$

moreover

$$\lim_n \mathcal{E}^{el}(t)(u^{\delta_n}_{\varepsilon_n,a}(t)) = \mathcal{E}^{el}(t)(u_a(t)); \tag{7.11}$$

(c) *we have*

$$\mathcal{E}^s(\Gamma_a(t)) \le \liminf_n \mathcal{E}^s(\Gamma^{\delta_n}_{\varepsilon_n,a}(t)).$$

Proof. We now perform a variant of [13, Theorem 4.7]. Let $(\varphi_k)_{k\in\mathbb{N}} \subseteq L^1(\Omega;\mathbb{R}^2)$ be dense in $L^1(\Omega;\mathbb{R}^2)$. For every φ_k and for every $m \in \mathbb{N}$, let $v^{n,a}_{k,m}(t)$ be a minimum of the problem

$$\min\{\|\nabla v\|_p + \|v\|_q + m\|v-\varphi_k\|_1 \,:\, v \in V^n_a\},$$

where

$$V^n_a := \{v \in \mathcal{A}^B_{\varepsilon_n,a}(\Omega;\mathbb{R}^2),\ S^{g^{\delta_n}_{\varepsilon_n}(t)}(v) \subseteq \Gamma^{\delta_n}_{\varepsilon_n,a}(t)\}.$$

Since by (7.4) we have $\mathcal{H}^1(\Gamma^{\delta_n}_{\varepsilon_n,a}(t)) \le C'$, by Theorem 2.1 there exists a subsequence of $(\delta_n,\varepsilon_n)_{n\in\mathbb{N}}$ (which we denote with the same symbol) such that $v^{n,a}_{k,m}(t)$ weakly converges to some $v^a_{k,m}(t) \in GSBV^p_q(\Omega;\mathbb{R}^2)$ as $n \to +\infty$ for all $k,m \in \mathbb{N}$. We set

$$\Gamma_a(t) := \bigcup_{k,m} S^{g(t)}(v^a_{k,m}(t)). \tag{7.12}$$

Let us see that $\Gamma_a(t)$ satisfies all the properties of the lemma. Clearly $\Gamma_a(t) \in \mathcal{R}(\overline{\Omega}_B;\partial_N\Omega)$ and point (c) is a consequence of Theorem 2.1. In particular by (7.4) we have that

$$\mathcal{H}^1(\Gamma_a(t)) \le C'. \tag{7.13}$$

Let us come to point (a). Let $w_n \in \mathcal{A}^B_{\varepsilon_n,a}(\Omega;\mathbb{R}^2)$ be such that $S^{g^{\delta_n}_{\varepsilon_n}(t)}(w_n) \subseteq \Gamma^{\delta_n}_{\varepsilon_n,a}(t)$ and

$$w_n \rightharpoonup w \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2).$$

We claim that there exists $k_m \to +\infty$ such that

$$v^a_{k_m,m}(t) \rightharpoonup w \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2). \tag{7.14}$$

Then since $S^{g(t)}(v^a_{k_m,m}(t)) \subseteq \Gamma_a(t)$ for all m and in view of (7.13), we deduce that $S^{g(t)}(w) \widetilde{\subset} \Gamma_a(t)$. Let us prove (7.14). Fixed $m \in \mathbb{N}$, let us choose k_m in such a way that

$$m\|w-\varphi_{k_m}\|_1 \to 0.$$

By minimality of $v^{n,a}_{k_m,m}(t)$ we have for all n

$$\|\nabla v^{n,a}_{k_m,m}(t)\|_p + \|v^{n,a}_{k_m,m}(t)\|_q + m\|v^{n,a}_{k_m,m}(t)-\varphi_{k_m}\|_1 \le \|\nabla w_n\|_p + \|w_n\|_q + m\|w_n-\varphi_{k_m}\|_1.$$

Passing to the limit in n, by lower semicontinuity we get for some $C \ge 0$

$$\|\nabla v^a_{k_m,m}(t)\|_p + \|v^a_{k_m,m}(t)\|_q + m\|v^a_{k_m,m}(t)-\varphi_{k_m}\|_1 \le C + m\|w-\varphi_{k_m}\|_1.$$

We deduce for $m \to +\infty$

$$\|v^a_{k_m,m}(t)-\varphi_{k_m}\|_1 \to 0,$$

which together with $\|\varphi_{k_m}-w\|_1 \to 0$ implies that

$$v^a_{k_m,m}(t) \to w \qquad \text{strongly in } L^1(\Omega;\mathbb{R}^2).$$

Since

$$\|\nabla v^a_{k_m,m}(t)\|_p + \|v^a_{k_m,m}(t)\|_q \le C + m\|w-\varphi_{k_m}\|_1 \le C+1$$

for m large, we have that $v^a_{k_m,m}(t) \rightharpoonup w$ weakly in $GSBV^p_q(\Omega;\mathbb{R}^2)$, and this proves (7.14).

Finally, let us come to point (b). Let $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ with $S(v) \subseteq \overline{\Omega}_B$, and let us fix $k_1,\dots,k_s$ and $m_1,\dots,m_r$ in $\mathbb{N}$. By Proposition 4.2, there exists $v_n \in \mathcal{A}^B_{\varepsilon_n,a}(\Omega;\mathbb{R}^2)$ such that

$$\lim_n \mathcal{E}^{el}(t)(v_n) = \mathcal{E}^{el}(t)(v)$$

and

$$\limsup_n \mathcal{E}^s\left(S^{g^{\delta_n}_{\varepsilon_n}(t)}(v_n)\setminus \Gamma^{\delta_n}_{\varepsilon_n,a}(t)\right) \le \limsup_n \mathcal{E}^s\big(S^{g^{\delta_n}_{\varepsilon_n}(t)}(v_n)\setminus \bigcup_{i\le s,\, j\le r} S(v^{n,a}_{k_i,m_j})\big)$$
$$\le \mu(a)\mathcal{E}^s\big(S^{g(t)}(v)\setminus \bigcup_{i\le s,\, j\le r} S(v^{a}_{k_i,m_j})\big),$$

where $\mu(a)\to 1$ as $a\to 0$. Since the k_i's and the m_j's are arbitrary, we obtain that

$$\limsup_n \mathcal{E}^s\left(S^{g^{\delta_n}_{\varepsilon_n}(t)}(v_n)\setminus \Gamma^{\delta_n}_{\varepsilon_n,a}(t)\right) \le \mu(a)\mathcal{E}^s\left(S^{g(t)}(v)\setminus \Gamma_a(t)\right). \tag{7.15}$$

Let us suppose that $u^{\delta_n}_{\varepsilon_n,a}(t)\rightharpoonup u_a(t)$ weakly in $GSBV^p_q(\Omega;\mathbb{R}^2)$ along a suitable subsequence which we indicate by the same symbol. By the minimality property (7.5), comparing $u^{\delta_n}_{\varepsilon_n,a}(t)$ with v_n we get

$$\mathcal{E}^{el}(t)(u^{\delta_n}_{\varepsilon_n,a}(t)) \le \mathcal{E}^{el}(t)(v_n) + \mathcal{E}^s\left(S^{g^{\delta_n}_{\varepsilon_n}(t)}(v_n)\setminus \Gamma^{\delta_n}_{\varepsilon_n,a}(t)\right) + o_n, \tag{7.16}$$

with $o_n\to 0$ as $n\to+\infty$. Then we have

$$\begin{aligned}\mathcal{E}^{el}(t)(u_a(t)) &\le \liminf_n \mathcal{E}^{el}(t)(u^{\delta_n}_{\varepsilon_n,a}(t))\\ &\le \limsup_n\left(\mathcal{E}^{el}(t)(v_n)+\mathcal{E}^s\left(S^{g^{\delta_n}_{\varepsilon_n}(t)}(v_n)\setminus \Gamma^{\delta_n}_{\varepsilon_n,a}(t)\right)\right)\\ &\le \mathcal{E}^{el}(t)(v)+\limsup_n \mathcal{E}^s\left(S^{g^{\delta_n}_{\varepsilon_n}(t)}(v_n)\setminus \Gamma^{\delta_n}_{\varepsilon_n,a}(t)\right)\\ &\le \mathcal{E}^{el}(t)(v)+\mu(a)\mathcal{E}^s\left(S^{g(t)}(v)\setminus \Gamma_a(t)\right),\end{aligned}$$

that is (7.10) holds. Choosing $v=u_a(t)$, passing to the limsup in (7.16), and taking into account (7.15) we obtain that

$$\limsup_n \mathcal{E}^{el}(t)(u^{\delta_n}_{\varepsilon_n,a}(t)) \le \mathcal{E}^{el}(t)(u_a(t)).$$

Since by (7.10) $\mathcal{E}^{el}(t)(u_a(t))$ is independent of the accumulation point $u_a(t)$, we conclude that (7.11) holds. □

Remark 7.3. Using Lemma 7.2, it is possible to construct an increasing family $\{t\to\Gamma_a(t) : t\in[0,T]\}$ and a subsequence of $(\delta_n,\varepsilon_n)_{n\in\mathbb{N}}$ such that points (a), (b) and (c) of Lemma 7.2 hold for every $t\in[0,T]$. This evolution $\{t\to\Gamma_a(t) : t\in[0,T]\}$ can be considered as an approximate quasistatic evolution, in the sense that it satisfies irreversibility, but satisfies static equilibrium and nondissipativity up to a small error due to the fact that a is kept fixed. The presence of $\mu(a)$ in the minimality property (7.10) takes into account the anisotropy in the approximation of the surface energy: in fact, since a is kept fixed, the adaptive edges of the triangulations $\mathcal{T}_{\varepsilon,a}(\Omega)$ cannot recover all the possible directions. The nondissipativity condition up to a small error can be obtained using the minimality property (7.10) and following [13, Theorem 3.13] (estimate from below of the total energy).

The construction of $\{t\to\Gamma_a(t) : t\in[0,T]\}$ is the following. If $D\subseteq[0,T]$ is countable and dense, by Lemma 7.2 and using a diagonalization argument, we can find a subsequence of $(\delta_n,\varepsilon_n)_{n\in\mathbb{N}}$ and an increasing family $\Gamma_a(t)\in\mathcal{R}(\overline{\Omega}_B;\partial_N\Omega)$, $t\in D$, such that points (a), (b) and (c) hold for every $t\in D$. Let us set for every $t\in[0,T]$

$$\Gamma^+_a(t) := \bigcap_{s\ge t,\, s\in D}\Gamma_a(s).$$

Clearly $\{t\to\Gamma^+_a(t) : t\in[0,T]\}$ is increasing, in the sense that $\Gamma_a(s)\,\tilde{\subset}\,\Gamma_a(t)$ for all $s\le t$. As a consequence, the set J of discontinuity points of $\mathcal{H}^1(\Gamma^+_a(t))$ is at most countable. We can extract a further subsequence of $(\delta_n,\varepsilon_n)_{n\in\mathbb{N}}$ such that $\Gamma_a(t)$ is determined also for all $t\in J$ (notice that $\Gamma_a(t)\,\tilde{\subset}\,\Gamma^+_a(t)$). For all $t\in[0,T]\setminus(D\cup J)$ we set $\Gamma_a(t):=\Gamma^+_a(t)$. We have that $\Gamma_a(t)\in\mathcal{R}(\overline{\Omega}_B;\partial_N\Omega)$ and $\{t\to\Gamma_a(t) : t\in[0,T]\}$ is increasing.

For $t \in D \cup J$, $\Gamma_a(t)$ satisfies by construction points (a), (b) and (c) of Lemma 7.2. Let us consider the case $t \in [0,T] \setminus (D \cup J)$.

Concerning point (a), we have that $S^{g(t)}(w) \,\widetilde{\subset}\, \Gamma_a(s)$ for every $s \in D \cap [t,T]$, so that passing to the intersection we get $S^{g(t)}(u_a(t)) \,\widetilde{\subset}\, \Gamma_a(t)$.

As for point (b), considering $s \in D \cap [0,t[$, for every $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ with $S(v) \,\widetilde{\subset}\, \overline{\Omega}_B$, we have there exists $v_n \in \mathcal{A}^B_{\varepsilon_n,a}(\Omega;\mathbb{R}^2)$ such that

$$\lim_n \mathcal{E}^{el}(t)(v_n) = \mathcal{E}^{el}(t)(v),$$

and

$$\limsup_n \mathcal{E}^s\left(S^{g^{\delta_n}_{\varepsilon_n}(t)}(v_n) \setminus \Gamma^{\delta_n}_{\varepsilon_n,a}(t)\right) \le \limsup_n \mathcal{E}^s\left(S^{g^{\delta_n}_{\varepsilon_n}(t)}(v_n) \setminus \Gamma^{\delta_n}_{\varepsilon_n,a}(s)\right)$$
$$\le \mu(a)\mathcal{E}^s\left(S^{g(t)}(v) \setminus \Gamma_a(s)\right).$$

Then by minimality property (7.5) and passing to the limit in n we have

$$\mathcal{E}^{el}(t)(u(t)) \le \mathcal{E}^{el}(t)(v) + \mu(a)\mathcal{E}^s\left(S^{g(t)}(v) \setminus \Gamma_a(s)\right).$$

Letting $s \to t$ we get that (7.10) holds. Reasoning as in Lemma 7.2, we get that also (7.11) holds.

Finally, coming to point (c), we have that for all $s \in D \cap [0,t[$

$$\liminf_n \mathcal{E}^s(\Gamma^{\delta_n}_{\varepsilon_n,a}(t)) \ge \liminf_n \mathcal{E}^s(\Gamma^{\delta_n}_{\varepsilon_n,a}(s)) \ge \mathcal{E}^s(\Gamma_a(s)),$$

so that letting $s \nearrow t$, and recalling that t is a continuity point for $\mathcal{E}^s(\Gamma^{\delta_n}_{\varepsilon_n,a}(\cdot))$, we obtain that the lower semicontinuity holds.

We can now let $a \to 0$.

Lemma 7.4. *There exist a map $\{t \to \Gamma(t) \in \mathcal{R}(\overline{\Omega}_B;\partial_N\Omega), t \in [0,T]\}$ and sequences $\delta_n \to 0$, $\varepsilon_n \to 0$, $a_n \to 0$ such that the following facts hold:*

(a) $\Gamma^0 \,\widetilde{\subset}\, \Gamma(s) \,\widetilde{\subset}\, \Gamma(t)$ *for all* $0 \le s \le t \le T$;

(b) *for all $t \in [0,T]$, if $w_n \in \mathcal{A}^B_{\varepsilon_n,a}(\Omega;\mathbb{R}^2)$ with $S^{g^{\delta_n}_{\varepsilon_n}(t)}(w_n) \subseteq \Gamma^{\delta_n}_{\varepsilon_n,a_n}(t)$ is such that*

$$w_n \rightharpoonup w \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2),$$

then we have

$$S^{g(t)}(w) \,\widetilde{\subset}\, \Gamma(t);$$

(c) *for all $t \in [0,T]$ and for every accumulation point $u(t)$ of $(u^{\delta_n}_{\varepsilon_n,a_n}(t))_{n\in\mathbb{N}}$ for the weak convergence in $GSBV^p_q(\Omega;\mathbb{R}^2)$ and for all $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ with $S(v) \,\widetilde{\subset}\, \overline{\Omega}_B$, we have*

$$\mathcal{E}^{el}(t)(u(t)) \le \mathcal{E}^{el}(t)(v) + \mathcal{E}^s\left(S^{g(t)}(v) \setminus \Gamma(t)\right), \tag{7.17}$$

and

$$\mathcal{E}^{el}(t)(u(t)) = \lim_n \mathcal{E}^{el}(t)(u^{\delta_n}_{\varepsilon_n,a_n}(t)); \tag{7.18}$$

(d) *for all $t \in [0,T]$ we have*

$$\mathcal{E}^s(\Gamma(t)) \le \liminf_n \mathcal{E}^s(\Gamma^{\delta_n}_{\varepsilon_n,a_n}(t)). \tag{7.19}$$

Proof. Let us consider $\delta_h \to 0$ and $\varepsilon_h \to 0$. Given $a \in]0, \frac{1}{2}[$ and $t \in [0,T]$, let $\Gamma_a(t)$ be the rectifiable set given by Lemma 7.2. Recall that by (7.12) we have

$$\Gamma_a(t) = \bigcup_{k,m} S^{g(t)}(v^a_{k,m}(t)),$$

where $v^a_{k,m}(t)$ is the weak limit in $GSBV^p_q(\Omega;\mathbb{R}^2)$ along a suitable subsequence depending on a of a minimum $v^{h,a}_{k,m}(t)$ of the problem

$$\min\{\|\nabla v\|_p + \|v\|_q + m\|v-\varphi_k\|_1 \,:\, v \in V^h_a(t)\}, \tag{7.20}$$

where $(\varphi_k)_{k\in\mathbb{N}} \subseteq L^1(\Omega;\mathbb{R}^2)$ is dense in $L^1(\Omega;\mathbb{R}^2)$ and

$$V^h_a(t) := \{v \in \mathcal{A}^B_{\varepsilon,a}(\Omega';\mathbb{R}^2), S^{g^{\delta_h}_{\varepsilon_h}(t)}(v) \subseteq \Gamma^{\delta_h}_{\varepsilon_h,a}(t)\}.$$

Let $a_n \to 0$, and let $D := \{t_j \,:\, j \in \mathbb{N}\} \subseteq [0,T]$ be countable and dense with $0 \in D$. Using a diagonal argument, up to a subsequence of $(\delta_h, \varepsilon_h)_{h\in\mathbb{N}}$, we may suppose that for all $t \in D$ and for all n

$$v^{h,a_n}_{k,m}(t) \rightharpoonup v^{a_n}_{k,m}(t) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2).$$

Moreover, we may assume that for all $t \in D$ and for all n

$$u^{\delta_h}_{\varepsilon_h,a_n}(t) \rightharpoonup u_{a_n}(t) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2)$$

with

$$\mathcal{E}^{el}(t)(u^{\delta_h}_{\varepsilon_h,a_n}(t)) \to \mathcal{E}^{el}(t)(u_{a_n}(t)).$$

By Lemma 7.2, we have that $u_{a_n}(t)$ satisfies the minimality property (7.10).

Up to a subsequence of $(a_n)_{n\in\mathbb{N}}$, we may suppose that for all k, m and $t \in D$ we have

$$v^{a_n}_{k,m}(t) \rightharpoonup v_{k,m}(t) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2), \tag{7.21}$$

and

$$u_{a_n}(t) \rightharpoonup u(t) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2). \tag{7.22}$$

For all $t \in D$, let us set

$$\Gamma(t) := \bigcup_{k,m} S^{g(t)}(v_{k,m}(t)). \tag{7.23}$$

By Proposition 4.2, in view of the minimality property (7.10) and taking into account that $\mu(a_n) \to 1$, we have that for all $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ with $S(v) \,\tilde{\subset}\, \overline{\Omega}_B$

$$\mathcal{E}^{el}(t)(u(t)) \le \mathcal{E}^{el}(t)(v) + \mathcal{E}^s(S^{g(t)}(v) \setminus \Gamma(t)), \tag{7.24}$$

and as a consequence, we obtain

$$\mathcal{E}^{el}(t)(u_{a_n}(t)) \to \mathcal{E}^{el}(t)(u(t)).$$

We now perform the following diagonal argument. Choose $\delta_{h_0}, \varepsilon_{h_0}$ in such a way that

$$\|v^{h_0,a_0}_{0,0}(t_0) - v^{a_0}_{0,0}(t_0)\|_1 + \|u^{\delta_{h_0}}_{\varepsilon_{h_0},a_0}(t_0) - u_{a_0}(t_0)\|_1 + |\mathcal{E}^{el}(t_0)(u^{\delta_{h_0}}_{\varepsilon_{h_0},a_0}(t_0)) - \mathcal{E}^{el}(t_0)(u_{a_0}(t_0))| \le 1.$$

Supposing to have constructed $\delta_{h_n}, \varepsilon_{h_n}$, we choose $\delta_{h_{n+1}}, \varepsilon_{h_{n+1}}$ in such a way that for all $k \le n+1$, $m \le n+1$ and for all t_i with $1 \le i \le n+1$ we have

$$\|v^{h_{n+1},a_{n+1}}_{k,m}(t_i) - v^{a_{n+1}}_{k,m}(t_i)\|_1 + \|u^{\delta_{h_{n+1}}}_{\varepsilon_{h_{n+1}},a_{n+1}}(t_i) - u_{a_{n+1}}(t_i)\|_1 + |\mathcal{E}^{el}(t_i)(u^{\delta_{h_{n+1}}}_{\varepsilon_{h_{n+1}},a_{n+1}}(t_i)) - \mathcal{E}^{el}(t_i)(u_{a_{n+1}}(t_i))| \le \frac{1}{n+1}.$$

Let us set $\delta_n := \delta_{h_n}$ and $\varepsilon_n := \varepsilon_{h_n}$, and let us prove that $\Gamma(t)$ defined in (7.23) satisfies the properties of the Lemma. We have immediately that $\Gamma(t) \in \mathcal{R}(\overline{\Omega}_B;\partial_N\Omega)$.

Concerning point (d), notice that

$$\Gamma^{\delta_n}_{\varepsilon_n,a_n}(t) = \bigcup_{m,k} S^{g^{\delta_n}_{\varepsilon_n}(t)}(v^{h_n,a_n}_{k,m}(t)), \qquad \Gamma(t) = \bigcup_{m,k} S^{g(t)}(v_{k,m}(t)),$$

and that for all k, m

$$v^{h_n,a_n}_{k,m}(t) \rightharpoonup v_{k,m}(t) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2);$$

then (7.19) is a consequence of Theorem 2.1. In particular, by (7.4), we get that

$$\mathcal{H}^1(\Gamma(t)) \le C'. \tag{7.25}$$

Let us come to point (b). Let $w_n \in \mathcal{A}^B_{\varepsilon_n,a}(\Omega;\mathbb{R}^2)$ with $S^{g^{\delta_n}_{\varepsilon_n}(t)}(w_n) \subseteq \Gamma^{\delta_n}_{\varepsilon_n,a_n}(t)$ be such that

$$w_n \rightharpoonup w \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2).$$

For every $m \in \mathbb{N}$, let us choose k_m in such a way that

$$m\|w - \varphi_{k_m}\|_1 \to 0.$$

By minimality of $v^{h_n,a_n}_{k_m,m}(t)$ we have for all n

$$\|\nabla v^{h_n,a_n}_{k_m,m}(t)\|_p + \|v^{h_n,a_n}_{k_m,m}(t)\|_q + m\|v^{h_n,a_n}_{k_m,m}(t) - \varphi_{k_m}\|_1 \le \|\nabla w_n\|_p + \|w_n\|_q + m\|w_n - \varphi_{k_m}\|_1.$$

By construction of h_n, and in view of (7.21), we have

$$v^{h_n,a_n}_{k_m,m}(t) \rightharpoonup v_{k_m,m}(t) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2).$$

Then passing to the limit in n, by lower semicontinuity we get for some $C \ge 0$

$$\|\nabla v_{k_m,m}(t)\|_p + \|v_{k_m,m}(t)\|_q + m\|v_{k_m,m}(t) - \varphi_{k_m}\|_1 \le C + m\|w - \varphi_{k_m}\|_1.$$

We deduce for $m \to +\infty$

$$\|v_{k_m,m}(t) - \varphi_{k_m}\|_1 \to 0,$$

which together with $\|\varphi_{k_m} - w\|_1 \to 0$ implies that

$$v_{k_m,m}(t) \to w \quad \text{strongly in } L^1(\Omega;\mathbb{R}^2).$$

Since

$$\|\nabla v_{k_m,m}(t)\|_p + \|v_{k_m,m}(t)\|_q \le C + m\|w - \varphi_{k_m}\|_1 \le C + 1$$

for m large, we have that

$$v_{k_m,m}(t) \rightharpoonup w \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2).$$

Since $S^{g(t)}(v_{k_m,m}(t)) \subseteq \Gamma(t)$ for all m, and since $\mathcal{H}^1(\Gamma(t)) < C'$, we deduce that $S^{g(t)}(w) \,\tilde{\subset}\, \Gamma(t)$.

Coming to point (c), we have that (7.18) holds by construction. Moreover (7.17) holds in view of (7.24) and by the fact that $u^{\delta_n}_{\varepsilon_n,a_n}(t)$ weakly converges in $GSBV^p_q(\Omega;\mathbb{R}^2)$ to $u(t)$ defined in (7.22).

In order to prove point (a), notice that if $s \le t$ with $s, t \in D$, we have for all $k, m \in \mathbb{N}$ that

$$S^{g^{\delta_n}_{\varepsilon_n}(t)}(v^{h_n,a_n}_{k,m}(s) + g^{\delta_n}_{\varepsilon_n}(t) - g^{\delta_n}_{\varepsilon_n}(s)) \subseteq \Gamma^{\delta_n}_{\varepsilon_n,a_n}(s) \subseteq \Gamma^{\delta_n}_{\varepsilon_n,a_n}(t),$$

and

$$v^{h_n,a_n}_{k,m}(s) + g^{\delta_n}_{\varepsilon_n}(t) - g^{\delta_n}_{\varepsilon_n}(s) \rightharpoonup v_{k,m}(s) + g(t) - g(s) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2),$$

where $v^{h,a}_{k,m}(s)$ and $v_{k,m}(s)$ are defined in (7.20) and (7.21). By point (b) we deduce that

$$S^{g(t)}(v_{k,m}(s) + g(t) - g(s)) = S^{g(s)}(v_{k,m}(s)) \,\tilde{\subset}\, \Gamma(t).$$

Then by the definition of $\Gamma(s)$ we get $\Gamma(s) \,\tilde{\subset}\, \Gamma(t)$. Finally, by the same argument, we deduce $\Gamma^0 \,\tilde{\subset}\, \Gamma(s)$.

In order to deal with all $t \in [0,T]$, we proceed as in Remark 7.3. For all $t \in [0,T] \setminus D$ let us set

$$\Gamma^+(t) := \bigcap_{s \ge t, s \in D} \Gamma(s).$$

Clearly $\Gamma^+(t) \in \mathcal{R}(\overline{\Omega}_B; \partial_N\Omega)$ and satisfies point (a), so that the set J of discontinuity points of $\mathcal{H}^1(\Gamma^+(\cdot))$ is at most countable. We can then extract a further subsequence of $(\delta_n, \varepsilon_n, a_n)_{n\in\mathbb{N}}$ such that $\Gamma(t)$ is determined also for all $t \in J \setminus D$ (notice that $\Gamma(t) \,\tilde{\subset}\, \Gamma^+(t)$). For all $t \in [0,T] \setminus (D \cup J)$ we set $\Gamma(t) := \Gamma^+(t)$. We have that $\Gamma(t) \in \mathcal{R}(\overline{\Omega}_B; \partial_N\Omega)$ and that $\Gamma(t)$ satisfies point (a). Let us see that $\Gamma(t)$ satisfies also points (b), (c) and (d) also for $t \in [0,T] \setminus (D \cup J)$.

Concerning point (b), for every accumulation point $u(t)$ of $(u^{\delta_n}_{\varepsilon_n,a_n}(t))_{n\in\mathbb{N}}$ for the weak convergence in $GSBV^p_q(\Omega;\mathbb{R}^2)$, by the first part of the proof, we have that $S^{g(t)}(u(t)) \,\tilde{\subset}\, \Gamma(s)$ for all $s \in D$ with $s \ge t$, so that passing to the intersection, we get that $S^{g(t)}(u(t)) \,\tilde{\subset}\, \Gamma(t)$.

Let us come to point (c). Let

$$u_j(t) := u^{\delta_{n_j}}_{\varepsilon_{n_j},a_{n_j}}(t) \rightharpoonup u(t) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2)$$

along a subsequence $n_j \nearrow +\infty$. Let us set $\Gamma_j := \Gamma^{\delta_{n_j}}_{\varepsilon_{n_j},a_{n_j}}$ and $g_j := g^{\delta_{n_j}}_{\varepsilon_{n_j}}$. Up to a further subsequence there exists $s_j \in D$ with $s_j \nearrow t$, and such that setting $u_j(s_j) := u^{\delta_{n_j}}_{\varepsilon_{n_j},a_{n_j}}(s_j)$, we have

$$\|u_j(s_j) - u(s_j)\|_1 + |\mathcal{E}^{el}(s_j)(u_j(s_j)) - \mathcal{E}^{el}(s_j)(u(s_j))| \to 0. \tag{7.26}$$

We have that there exists $u^*(t) \in GSBV^p_q(\Omega;\mathbb{R}^2)$ such that up to a subsequence

$$u(s_j) \rightharpoonup u^*(t) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2).$$

By the minimality property (7.17) of $u(s_j)$, for all $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ with $S(v) \,\widetilde{\subset}\, \overline{\Omega}_B$, we have that

$$\mathcal{E}^{el}(s_j)(u(s_j)) \le \mathcal{E}^{el}(s_j)(v - g(t) + g(s_j)) + \mathcal{E}^s(S^{g(t)}(v) \setminus \Gamma(s_j)).$$

Passing to the limit in j we have that for all $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ with $S(v) \,\widetilde{\subset}\, \overline{\Omega}_B$

$$\mathcal{E}^{el}(t)(u^*(t)) \le \mathcal{E}^{el}(t)(v) + \mathcal{E}^s(S^{g(t)}(v) \setminus \Gamma(t)). \tag{7.27}$$

As a consequence of the stability of this unilateral minimality property, it follows that

$$\mathcal{E}^{el}(s_j)(u(s_j)) \to \mathcal{E}^{el}(t)(u^*(t)).$$

By (7.26) we get

$$u_j(s_j) \rightharpoonup u^*(t) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2),$$

and

$$\mathcal{E}^{el}(s_j)(u_j(s_j)) \to \mathcal{E}^{el}(t)(u^*(t)). \tag{7.28}$$

By (7.5), comparing $u_j(t)$ with $u_j(s_j) - g_j(s_j) + g_j(t)$, taking into account that

$$S^{g_j(s_j)}(u_j(s_j)) \subseteq \Gamma_j(s_j) \subseteq \Gamma_j(t),$$

we obtain

$$\mathcal{E}^{el}(t)(u_j(t)) \le \mathcal{E}^{el}(s_j)(u_j(s_j)) + o_j$$

where $o_j \to 0$ as $j \to +\infty$. Passing to the limit in j we have by (7.28)

$$\mathcal{E}^{el}(t)(u(t)) \le \liminf_j \mathcal{E}^{el}(t)(u_j(t)) \le \limsup_j \mathcal{E}^{el}(t)(u_j(t)) \le \mathcal{E}^{el}(t)(u^*(t)).$$

By (7.27) we deduce that (7.17) holds. Moreover we have that $\mathcal{E}^{el}(t)(u(t)) = \mathcal{E}^{el}(t)(u^*(t))$ and that $\mathcal{E}^{el}(t)(u(t))$ is independent of the accumulation point $u(t)$. Then we deduce that (7.18) holds.

Finally, concerning point (d), we have that for all $s \in D \cap [0,t[$

$$\liminf_n \mathcal{E}^s(\Gamma^{\delta_n}_{\varepsilon_n,a_n}(t)) \ge \liminf_n \mathcal{E}^s(\Gamma^{\delta_n}_{\varepsilon_n,a_n}(s)) \ge \mathcal{E}^s(\Gamma(s)),$$

so that letting $s \nearrow t$ we obtain (7.19). The proof is now complete. □

We can now prove Theorem 7.1.

<ins>PROOF OF THEOREM 7.1</ins>. Let $(\delta_n, \varepsilon_n, a_n)_{n\in\mathbb{N}}$ and $\{t \to \Gamma(t) \in \mathcal{R}(\overline{\Omega}_B;\partial_N\Omega), t \in [0,T]\}$ be given by Lemma 7.4. For all $t \in [0,T]$, let us set

$$u_n(t) := u^{\delta_n}_{\varepsilon_n,a_n}(t), \qquad \Gamma_n(t) := \Gamma^{\delta_n}_{\varepsilon_n,a_n}(t).$$

Let us see that it is possible to choose an accumulation point $u(t) \in GSBV^p_q(\Omega;\mathbb{R}^2)$ of $(u_n(t))_{n\in\mathbb{N}}$ such that $\{t \to (u(t),\Gamma(t)) : t \in [0,T]\}$ is a quasistatic growth of brittle fractures in the sense of Dal Maso-Francfort-Toader. Let us set

$$\begin{aligned}\vartheta_n(s) := \langle \partial\mathcal{W}(\nabla u_n(s)), \nabla \dot g_{\varepsilon_n}(s)\rangle \\ - \dot{\mathcal{F}}(s)(u_n(s)) - \langle \partial\mathcal{F}(s)(u_n(s)), \dot g_{\varepsilon_n}(s)\rangle \\ - \dot{\mathcal{G}}(s)(u_n(s)) - \langle \partial\mathcal{G}(s)(u_n(s)), \dot g_{\varepsilon_n}(s)\rangle.\end{aligned}$$

By growth conditions of $\mathcal{W}, \mathcal{F}, \mathcal{G}$ and by (7.4) we have that there exists $\psi \in L^1(0,T)$ such that $\vartheta_n(s) \leq \psi(s)$ for all n. Let us consider

$$\vartheta(s) := \limsup_n \vartheta_n(s).$$

By [13, Theorem 5.5 and Lemma 4.11], for every $s \in [0,T]$ there exists $u(s)$ accumulation point of $(u_n(s))_{n\in\mathbb{N}}$ for the weak convergence in $GSBV^p_q(\Omega;\mathbb{R}^2)$ such that

$$\begin{aligned}\vartheta(s) := \langle \partial\mathcal{W}(\nabla u(s)), \nabla \dot g(s)\rangle \\ - \dot{\mathcal{F}}(s)(u(s)) - \langle \partial\mathcal{F}(s)(u(s)), \dot g(s)\rangle \\ - \dot{\mathcal{G}}(s)(u(s)) - \langle \partial\mathcal{G}(s)(u(s)), \dot g(s)\rangle.\end{aligned}$$

Applying Fatou's Lemma (in the limsup version) to (7.6) with $s = 0$, we have that

$$\mathcal{E}(t)(u(t),\Gamma(t)) \leq \limsup_n \mathcal{E}(0)(u_n(0),\Gamma_n(0)) + \int_0^t \vartheta(s)\,ds.$$

By Proposition 5.1, we have that $\limsup_n \mathcal{E}(0)(u_n(0),\Gamma_n(0)) = \mathcal{E}(0)(u(0),\Gamma(0))$, so that we get

$$\mathcal{E}(t)(u(t),\Gamma(t)) \leq \mathcal{E}(0)(u(0),\Gamma(0)) + \int_0^t \vartheta(s)\,ds.$$

Moreover, again by [13, Theorem 3.13],

$$\mathcal{E}(t)(u(t),\Gamma(t)) \geq \mathcal{E}(0)(u(0),\Gamma(0)) + \int_0^t \vartheta(s)\,ds,$$

so that

$$\mathcal{E}(t)(u(t),\Gamma(t)) = \mathcal{E}(0)(u(0),\Gamma(0)) + \int_0^t \vartheta(s)\,ds. \tag{7.29}$$

We deduce that $\{t \to (u(t),\Gamma(t)) \,:\, t \in [0,T]\}$ is a quasistatic growth of brittle fractures: in fact by Lemma 7.4 we get that $\Gamma(\cdot)$ is increasing, and for $t \in [0,T]$ $(u(t),\Gamma(t)) \in AD(g(t))$ and the static equilibrium holds; moreover the nondissipativity condition is given by (7.29).

Let us see that points (a) and (b) of Theorem 7.1 holds. By (7.4), $(u_n(t))_{n\in\mathbb{N}}$ is weakly precompact in $GSBV^p_q(\Omega;\mathbb{R}^2)$ for all $t \in [0,T]$. Moreover by Lemma 7.4 every accumulation point $\tilde u(t)$ of $(u_n(t))_{n\in\mathbb{N}}$ for the weak convergence in $GSBV^p_q(\Omega;\mathbb{R}^2)$ is such that $S^{g(t)}(\tilde u(t)) \subseteq \Gamma(t)$ and the minimality property (7.7) holds. Moreover we have

$$\mathcal{E}^{el}(t)(\tilde u(t)) = \lim_n \mathcal{E}^{el}(t)(u_n(t)).$$

Since $\mathcal{E}^{el}(t)(\tilde u(t))$ is independent of the particular accumulation point $\tilde u(t)$, we have that point (a) is proved.

Let us come to point (b). Taking into account (7.18) and (7.19), for all $t \in [0,T]$ we have

$$E(t) \leq \liminf_n E_n(t) \leq \limsup_n E_n(t) \leq E(0) + \int_0^t \vartheta(s)\,ds = E(t),$$

so that (7.8) holds. Moreover we deduce that separate convergence of elastic and surface energies holds at any time, so that (7.9) is proved. The proof is now concluded. □

8. The strictly convex case

In this section we assume that the function $W(x,\xi)$ is strictly convex in ξ for a.e. $x \in \Omega$ and that the function $F(t,x,z)$ is strictly convex in z for all $t \in [0,T]$ and for a.e. $x \in \Omega$: as a consequence, the elastic energy $\mathcal{E}^{el}(t,v)$ is strictly convex in v for all $t \in [0,T]$, and a stronger approximation result is available.

Theorem 8.1. *Let $g \in W^{1,1}\big([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)\big)$ and let*

$$g_\varepsilon \in W^{1,1}\big([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)\big), \qquad g_\varepsilon(t) \in \mathcal{AF}_\varepsilon(\Omega;\mathbb{R}^2) \quad \text{for all } t \in [0,T]$$

be such that for $\varepsilon \to 0$

$$g_\varepsilon \to g \qquad \text{strongly in } W^{1,1}([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)).$$

Let $\Gamma^0 \in \mathbf{\Gamma}(\Omega)$ be an initial crack and let $\Gamma^0_{\varepsilon,a}$ be its approximation in the sense of Proposition 5.1. Let us suppose that

$$\begin{aligned} &W(x,\cdot) \text{ is strictly convex for a.e. } x \in \Omega,\\ &F(t,x,\cdot) \text{ is strictly convex for a.e. } (t,x) \in [0,T] \times \Omega. \end{aligned} \tag{8.1}$$

Given $\delta > 0$, $\varepsilon > 0$, $a \in]0, \frac{1}{2}[$, let $\big\{t \to \big(u^\delta_{\varepsilon,a}(t), \Gamma^\delta_{\varepsilon,a}(t)\big) : t \in [0,T]\big\}$ be the piecewise constant interpolation of the discrete evolution given by Proposition 6.1 relative to the initial crack $\Gamma^0_{\varepsilon,a}$ and the boundary data g_ε. Then there exists a quasistatic evolution $\{t \to (u(t), \Gamma(t)) : t \in [0,T]\}$ relative to the initial crack Γ^0 and the boundary data g in the sense of Theorem 3.1, and sequences $\delta_n \to 0$, $\varepsilon_n \to 0$, $a_n \to 0$, such that setting

$$u_n(t) := u^{\delta_n}_{\varepsilon_n,a_n}(t), \qquad \Gamma_n(t) := \Gamma^{\delta_n}_{\varepsilon_n,a_n}(t),$$

for all $t \in [0,T]$ the following facts hold:

(a) *$\nabla u_n(t) \to \nabla u(t)$ strongly in $L^p(\Omega; M^{2\times 2})$ and $u_n(t) \to u(t)$ strongly in $L^q(\Omega;\mathbb{R}^2)$;*

(b) *$\mathcal{E}(t)(u_n(t), \Gamma_n(t)) \to \mathcal{E}(t)(u(t), \Gamma(t))$, and in particular elastic and surface energies converge separately, that is*

$$\mathcal{E}^{el}(t)(u_n(t)) \to \mathcal{E}^{el}(t)(u(t)), \qquad \mathcal{E}^s(\Gamma_n(t)) \to \mathcal{E}^s(\Gamma(t)).$$

Proof. Let us consider the sequence $(\delta_n, \varepsilon_n, a_n)_{n\in\mathbb{N}}$ and the quasistatic growth of brittle fractures $\{t \to (u(t), \Gamma(t)) : t \in [0,T]\}$ given in Theorem 7.1. Under assumptions (8.1), we have that $u(t)$ is uniquely determined, because by (7.7) $u(t)$ minimizes

$$\min\{\mathcal{E}^{el}(t)(v) : v \in GSBV^p_q(\Omega;\mathbb{R}^2),\ S^{g(t)}(v) \subseteq \Gamma(t)\},$$

and $\mathcal{E}^{el}(t)(\cdot)$ is strictly convex. We conclude by point (a) of Theorem 7.1 that $u_n(t) \rightharpoonup u(t)$ weakly in $GSBV^p_q(\Omega;\mathbb{R}^2)$. Point (b) is a direct consequence of Theorem 7.1. By the convergence of the elastic energy, we deduce that

$$\begin{aligned} \lim_n \int_\Omega W(x, \nabla u_n(t))\,dx &= \int_\Omega W(x, \nabla u(t))\,dx,\\ \lim_n \int_\Omega F(t,x,u_n(t))\,dx &= \int_\Omega F(t,x,u(t))\,dx. \end{aligned}$$

By the assumption on the strict convexity of W and F we deduce by [7]

$$\nabla u_n(t) \to \nabla u(t) \qquad \text{strongly in } L^p(\Omega; M^{2\times 2}),$$

and

$$u_n(t) \to u(t) \qquad \text{strongly in } L^q(\Omega;\mathbb{R}^2).$$

Point (a) is now proved, and the proof is concluded. □

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